Structural approximation

Boris Zilber
University of Oxford

March 5, 2010

1 Introduction

1.1 Our motivation for the work presented below comes from the realisation of the rather paradoxical situation with the mathematics used by physicists in the last 70 or so years. Physicists have always been ahead of mathematicians in introducing and testing new methods of calculations, leaving to mathematicians the task of putting the new methods and ideas on a solid and rigorous foundation. But this time, with developments in quantum field theory huge progress achieved by physicists in dealing with singularities and non-convergent sums and integrals (famous Feynman path integrals) has not been matched so far, after all these years, with an adequate mathematical theory. A nice account of some of these methods with a demonstration of challenging calculations can be found in [4].

One may suggest that the success in carrying out these calculations in absence of a rigorous mathematical theory is due to the fact that the physicist, in fact, uses an implicit or explicit knowledge of the structure of his model which is not yet available in explicit mathematical terms. (A beautiful example of an honest attempt by a mathematician to decipher physicists' jargon provides the introductory section of [9].) In particular, many formulas which to the mathematician's eye are defined in terms of a metric or a measure are not what they look. Typically, it is crucial that discrete approximations to the continuous models have the same type of symmetries or other structural properties, but the formula does not say this.

1.2 Model theory and logic in general has an obvious advantage over general mathematics in this situation. The logician is not restricted by any conventional mathematics and is ready to deal with any type of structure at all. Moreover, modern model theory has worked out, in fact, a very efficient hierarchy of types of structures (stability theory and beyond), and has a crucial experience in introducing new tailor-made structures to accommodate and deal with specific mathematical problems. One particularly relevant class of structures was discovered by Hrushovski and the author [3] in an attempt to identify and characterise, essentially, "logically perfect" structures, or more technically, the top of the stability hierarchy. These are *Zariski structures* (in some variation called also *Zariski geometries*), defined in very general terms of geometric flavour and modelled on algebraic varieties over algebraically closed fields equipped with relations corresponding to Zariski closed sets.

The present author discovered that there is another important source of Zariski geometries – quantum algebras satisfying certain assumptions give rise to Zariski geometries in the same way as commutative affine algebras correspond to affine algebraic varieties. One class of such geometric objects, corresponding to quantum 2-tori T_q^2 is studied in this paper. In particular we are interested in the deformation of these structures as q varies.

1.3 The process of understanding the physical reality by working in an **ideal** model $\mathbf{M}_{\text{ideal}}$ can be interpreted as follows. We assume that the ideal model $\mathbf{M}_{\text{ideal}}$ is being chosen from a class of "nice" structures, which allows a good theory. We suppose that the real structure $\mathbf{M}_{\text{real}}$ is "very similar" to $\mathbf{M}_{\text{ideal}}$, meaning by this that the description of $\mathbf{M}_{\text{ideal}}$ (as a set of statements) has a large intersection with the corresponding description of M_{real}. The notion of "large" is of course relative and can be formalised dynamically, by assuming that $\mathbf{M}_{\text{ideal}}$ is approximated by a sequence $\mathbf{M}_i$ of structures and $\mathbf{M}_{\text{real}}$ is one of these, $\mathbf{M}_i = \mathbf{M}_{\text{real}}$, sufficiently close to $\mathbf{M}_{\text{ideal}}$. The notion of approximation must also contain both logical (qualitative) and topological ingredients. Topology gives us a way to speak of "nearness" between points and events. Naturally, the reason that we wouldn't distinguish two points in the ideal model $\mathbf{M}_{\text{ideal}}$ is that the corresponding points are very close in the real world $\mathbf{M}_{\text{real}}$, so that we do not see the difference (using the tools available). In the limit of the $\mathbf{M}_i$'s this sort of difference will manifest itself as an infinitesimal. In other words, the limit passage from the sequence $\mathbf{M}_i$ to the ideal model $\mathbf{M}_{\text{ideal}}$ must happen by killing the infinitesimal differences. This corresponds to taking a specialisation ("equations" preserving map) from an ultraproduct $\prod_D \mathbf{M}_i$ to $\mathbf{M}_{\text{ideal}}$.

We formalise and study the notion of a *structural approximation* below. We give a number of examples that demonstrate that this notion covers some well-known notions of approximation such as:

- limit point in a topological space;
- Gromov-Hausdorff limit of metric spaces;
- deformation of algebraic varieties

We also consider carefully an example of approximation that we think is a true version of a *quantum deformation.*

Note that the scheme is quite delicate regarding metric issues. In principle we may have a well-defined metric, agreeing with the qualitative topology, on the ideal structure only. Existence of a metric, especially the one that gives rise to a structure of a differentiable manifold, is one of the key reasons of why we regard some structures as 'nice' or 'tame'. The problem of whether and when a metric on M can be passed to approximating structures M_i might be difficult, indeed we don't know how to answer this problem in some interesting cases.

2 Definitions

2.1 General scheme of structural approximation

Following [1] by a **topological language** we mean a relational language $\mathcal{C}$ which will always be interpreted so that any n-ary $P \in \mathcal{C}$ (basic (primitive) $\mathcal{C}$-predicate) defines a closed subset $P(\mathbf{M})$ of M^n in any $\mathcal{C}$-structure $\mathbf{M}$, in the sense of a topology on M^n, all $n \in \mathbb{N}$. Not every closed subset of the topology in question is necessarily assumed to have the form $P(\mathbf{M})$, so those which are will be called **$\mathcal{C}$-closed**.

We assume that the equality is closed and all structures in question satisfy the $\mathcal{C}$-theory which ascertains that

- if $S_i \in \mathcal{C}$, $i = 1, 2$, then $S_1 \& S_2 \equiv P_1$, $S_1 \vee S_2 \equiv P_2$, for $P_1, P_2 \in \mathcal{C}$;
- if $S \in \mathcal{C}$, then $\forall x S \equiv P$, for some $P \in \mathcal{C}$;

We say that a $\mathcal{C}$-structure $\mathbf{M}$ is **complete** if, for each $S(x,y) \in \mathcal{C}$ there is $P(y) \in \mathcal{C}$ such that $\mathbf{M} \vDash \exists x S \equiv P$.

Note that we can always make $\mathbf{M} = (M, \mathcal{C})$ complete by extending $\mathcal{C}$ with relations corresponding to $\exists x S$ for all S in the original $\mathcal{C}$. We will call such an extension of the topology **the trivial completion** of $\mathbf{M}$.

We say $\mathbf{M}$ is **quasi-compact** (often just **compact**) if $\mathbf{M}$ is complete, every point in M is closed and for any filter of closed subsets of M^n the intersection is nonempty.

Remark The family of $\mathcal{C}$-closed sets forms a basis of a topology, the closed sets of which are just the infinite interestions of filters of $\mathcal{C}$-closed sets (the topology generated by $\mathcal{C}$).

If the topology generated by $\mathcal{C}$ is Noetherian then its closed sets are exactly the ones which are $\mathcal{C}$-closed.

Definition Given a structure $\mathbf{M}$ in a topological language $\mathcal{C}$ and structures $\mathbf{M}_i$ in the same language we say that $\mathbf{M}$ is **approximated** by $\mathbf{M}_i$ along an ultrafilter D if for some $\mathbf{M}^D \succcurlyeq \prod_D \mathbf{M}_i$ there is a surjective homomorphism

$$\lim : \mathbf{M}^D \to \mathbf{M}.$$

2.2 Proposition. *Suppose every point of* $\mathbf{M}$ *is closed and* $\mathbf{M}$ *is approximated by the sequence* $\{\mathbf{M}_i = \mathbf{M} : i \in I\}$ *for some* I *along an ultrafilter* D *on* I*, such that* $\mathbf{M}^D$ *is saturated. Then the trivial completion of* $\mathbf{M}$ *is quasi-compact.*

Proof Consider the $\mathbf{M}_i$ and $\mathbf{M}$ trivially completed, that is in the extended toplogy. Note that the given $\lim : \mathbf{M}^D \to \mathbf{M}$ is still a homomorphism in this language, since a homomorphism preserves positive formulas.

Closedness of points means that for every $a \in \mathbf{M}$ there is a positive one-variable $\mathcal{C}$-formula P_a with the only realisation a in $\mathbf{M}$. Under the assumptions for ${}^*\mathbf{M} \succ \mathbf{M}$, setting for $a \in \mathbf{M}$, $i(a)$ to be the unique realisation $\hat{a} \in {}^*\mathbf{M}$ of P_a we get an elementary embedding $i : \mathbf{M} \prec {}^*\mathbf{M}$. Now lim becomes a specialisation onto $\mathbf{M}$. This implies by [5] (see also a proof in [1]) that $\mathbf{M}$ is quasi-compact. $\Box$

Corollary – Assumption According to the proposition we will normally consider only approximations to quasi-compact structures.

2.3 Proposition *Suppose* $\mathbf{M}$ *is a quasi-compact topological* $\mathcal{C}$*-structure and* $\mathbf{N}$ *is an* $|M|$*-saturated* $\mathcal{C}$*-structure such that* $\mathbf{N}$ *is complete and for every positive* $\mathcal{C}$*-sentence* σ

$$\mathbf{N} \vDash \sigma \Rightarrow \mathbf{M} \vDash \sigma.$$

Then there is a surjective homomorphism $\lim : \mathbf{N} \to \mathbf{M}$.

Proof Given $A \subseteq N$, a partial *strong homomorphism* $\lim_A : A \to \mathbf{M}$ is a map defined on A such that for every $a \in A^k$, $\hat{a} = \lim_A a$ and $S(x,y) \in \mathcal{C}$ such that $\mathbf{N} \vDash \exists y S(a,y)$, we have $\mathbf{M} \vDash \exists y\, S(\hat{a},y)$.

When $A = \emptyset$ the map is assumed empty but the condition still holds, for any sentence of the form $\exists y\, S(y)$. So it follows from our assumptions that $\lim_\emptyset$ does exist.

Claim 1. Suppose for some $A \subseteq N$ there is a partial strong homomorphism $\lim_A : A \to \mathbf{M}$, and $b \in N$. Then $\lim_A$ can be extended to a partial strong homomorphism $\lim_{Ab} : Ab \to \mathbf{M}$.

Proof of Claim. Let $\mathbf{N} \vDash \exists z\, S(a,b,z)$, for $S(x,y,z)$ a positive formula and a a tuple in $\mathbf{N}$. Then $\mathbf{N} \vDash \exists yz\, S(a,y,z))$ and hence $\mathbf{M} \vDash \exists yz S(\hat{a},y,z))$.

It follows that the family of closed sets in $\mathbf{M}$ defined by $\{\exists z S(\hat{a},y,z) : \mathbf{N} \vDash \exists z\, S(a,b,z)\}$ is a filter. By quasi-compactness of $\mathbf{M}$ there is a point, say $\hat{b}$ in the intersection. Clearly, letting $\lim_{Ab} : b \to \hat{b}$, we preserve formulas of the form $\exists z\, S(x,y,z)$. Claim proved.

Claim 2. For $A \subset N$, $|A| \leq |M|$, assume $\lim_A$ exists and let $\hat{b} \in M \setminus A$. Then there is a $b \in N$ and an extension $\lim_{Ab} : b \mapsto \hat{b}$.

Proof. Consider the type over A,

$$p = \{\neg\exists z\, S(a,y,z) : \mathbf{M} \vDash \neg\exists z\, S(\hat{a},\hat{b},z) : \hat{a} = \lim_A a, a \subset A, S \in \mathcal{C}\}.$$

This is consistent in $\mathbf{N}$ since otherwise

$$\mathbf{N} \vDash \forall y \bigvee_{i=1}^{k} \exists z_i\, S_i(a,y,z_i)$$

for some finite subset of the type. The formula on the right is equivalent to $P(a)$, some $P \in \mathcal{C}$, so

$$\mathbf{M} \vDash \forall y \bigvee_{i=1}^{k} \exists z_i\, S_i(\hat{a},y,z_i)$$

$$\mathbf{M} \vDash \bigvee_{i=1}^{k} \exists z_i\, S_i(\hat{a},\hat{b},z_i),$$

the contradiction. Claim proved.

In order to prove the proposition consider a maximal partial strong homomorphism $\lim = \lim_A : A \to M$. By Claim 1 $A = N$, so lim is a total map on N. By Claim 2 lim is surjective.□

2.4 Proposition *Suppose every point of* $\mathbf{M}$ *is closed and* $\mathbf{M}$ *is approximated by* $\mathbf{M}_i$ *along* D, $\mathbf{M}^D \equiv \prod_D \mathbf{M}_i$ *is* $|M|^+$*-saturated,* $\lim : \mathbf{M}^D \to \mathbf{M}$ *and for every* $P \in \mathcal{C}$ *and every* $c \in M^m$

$$\mathbf{M} \vDash P(c) \Rightarrow \exists c' \lim c' = c \quad \textit{and}\ \mathbf{M}^D \vDash P(c').$$

Then there is an inverse embedding

$$\mathrm{colim} : \mathbf{M} \to \mathbf{M}^D, \quad \lim \mathrm{colim} = \mathrm{id}.$$

Proof Consider the set of variables $\{x_a : a \in M\}$ and the collection of atomic formulas

$$\Phi = \{P(x_{a_1}, \ldots, x_{a_n}) : a_1, \ldots, a_n \in M,\ \mathbf{M} \vDash P(a_1, \ldots, a_n),\ P \ \text{ atomic}\}.$$

We claim that Φ is consistent in $\mathbf{M}^D$. Indeed, by the assumption of the Proposition we can interpret $x_{a_1}, \ldots, x_{a_n}$ in M^D as some $a'_1, \ldots, a'_n$ such that $\lim a'_i = a_i$, $i = 1, \ldots, n$.

Now, by saturation, we can realise Φ in $\mathbf{M}^D$, say by elements $\{\hat{a} : a \in M\}$. Then $\mathbf{M}^D \vDash P_a(\hat{a})$ and so $\lim \hat{a} = a$.

Now, for any atom P

$$\mathbf{M}^D \vDash P(\hat{a}_1, \ldots, \hat{a}_n) \Leftrightarrow \mathbf{M} \vDash P(a_1, \ldots, a_n).$$

Indeed, the left arrow follows from Φ and the right one from the fact that $\lim \hat{a} = a$.

Hence the map $\mathrm{colim} : a \mapsto \hat{a}$ is an embedding $\mathbf{M} \to \mathbf{M}^D$.□

2.5 The assumptions in the Proposition above are quite strong. In practice we may be interested in a weaker condition which motivates the following.

Definition Given a structure $\mathbf{M}$ in a topological language and structures $\mathbf{M}_i$ in the same language we say that $\mathbf{M}$ is **strongly approximated** by $\mathbf{M}_i$ along an ultrafilter D if $\mathbf{M}$ is approximated in the topological language $\mathcal{C}$,

$$\lim : \mathbf{M}^D \to \mathbf{M},$$

and there is an $\mathcal{C}$-embedding

$$\mathrm{colim} : \mathbf{M} \to \mathbf{M}^D, \quad \lim \circ \mathrm{colim} = \mathrm{id}.$$

Suppose $\mathbf{M}$ is strongly approximated by $\mathbf{M}_i$ and we are given maps ψ_i on $\mathbf{M}_i$. Extending the language by the relation symbol $P(x, y)$ corresponding to the graph of ψ_i in $\mathbf{M}_i$ we get a function $\psi : \mathbf{M}^D \to \mathbf{M}^D$ given by $\psi(x) = y \leftrightarrow P(x, y)$. Now we can define for $w \in M$, the 'limit'

$$\psi(w) := \lim(\psi(\mathrm{colim}\, w)) \in M$$

This defines a new map $\psi : M \mapsto M$. Obviously we have

$$P(\mathbf{M}) \subseteq \lim P(\mathbf{M}^D).$$

Note that ψ does not depend on the sequence and on lim in case $P \in \mathcal{C}$.

2.6 Definition Perfect approximation: This is stronger than strong approximation by requiring that

$$\mathrm{colim} : \mathbf{M} \underset{\to}{\preceq} \mathbf{M}^D$$

is an elementary embedding.

Remark The cited above theorem of Weglorz implies that any quasi-compact $\mathbf{M}$ is perfectly approximated by any sequence $\{\mathbf{M}_i = \mathbf{M} : i \in I\}$.

3 Examples

In this section we assume for simplicity that $\mathbf{M}^D = \prod \mathbf{M}_i/D$.

3.1 Metric spaces

Let $\mathbf{M}$ and $\mathbf{M}_i$ be metric spaces in the language of binary predicates $d_r^{\leq}(x,y)$ and $d_r^{\geq}(x,y)$, all $r \in \mathbb{Q}$, $\ r \geq 0$, with the interpretation $\mathrm{dist}(x,y) \leq r$ and $\mathrm{dist}(x,y) \geq r$ correspondingly. The sets given by positive existential formulas in this language form our class $\mathcal{C}$.

Proposition. Assume $\mathbf{M}$ is compact and

$$\mathbf{M} = \text{GH-lim}_D \mathbf{M}_i,$$

the Gromov-Hausdorff limit of metric spaces along a non-principal ultrafilter D on I. Then

$$\mathbf{M} = \lim_D \mathbf{M}_i$$

Proof By definition, for any n there is an $X_n \in D$ such that $\mathrm{dist}(M_i, M) \leq \frac{1}{n}$, in a space containing both all the M_i for $i \in X_n$ and M. For any $\alpha \in \prod_i M_i$ define $\hat{\alpha}$ to be an element of M^I such that $\hat{\alpha}(i)$ is an element of M at a minimal distance from $\alpha(i)$ (choose one if there is more than one at the minimal distance). Let a_α be the limit point of the sequence $\{\hat{\alpha}(i) : i \in I\}$ along D in M. We define

$$\lim_D \alpha := a_\alpha.$$

It follows from the construction that, for $\alpha, \beta \in \prod_i M_i$,

$$\{i \in I : M_i \vDash d_r(\alpha(i), \beta(i))\} \in D \Rightarrow M \vDash d_r(\lim_D \alpha, \lim_D \beta).$$

We are not aware of any existing direct analogue of the Gromov-Hausdorff limit for topological spaces but the examples below demonstrate that the structural approximation surves this purpose well.

3.2 Cyclic groups in the profinite topology

Consider the coset-topology on $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$. The compactification of $\mathbb{Z}$ is then $\hat{\mathbb{Z}}$, the profinite completion. Choose a non-principal ultrafilter D on $\mathbb{N}$ so that $m\mathbb{N} \in D$ for every positive integer m (a *profinite ultrafilter*).

Claim.

$$\prod_D \mathbb{Z}/n\mathbb{Z} \cong \hat{\mathbb{Z}} \dot{+} \mathbb{Q}^\kappa \dot{+} T, \text{ some cardinal } \kappa \text{ and the torsion subgroup } T.$$

Proof Follows from the Eklof-Fisher classification of saturated models of Abelian groups.

Now define $\lim : \hat{\mathbb{Z}} \dot{+} \mathbb{Q}^\kappa \dot{+} T \to \hat{\mathbb{Z}}$ to be the projection (with kernel $\mathbb{Q}^\kappa \dot{+} T$) and colim the obvious embedding $\hat{\mathbb{Z}} \to \hat{\mathbb{Z}} \dot{+} \mathbb{Q}^\kappa \dot{+} T$, so

Proposition 1 *$\hat{\mathbb{Z}}$ is strongly approximated by $\mathbb{Z}/n\mathbb{Z}$ in the profinite topology.*

As an example, consider the element (sequence) $\gamma(n)$ such that $\gamma(n) = \frac{n}{2} \bmod n$, all $n \in 2\mathbb{N}$. Then $2\gamma = 0$ in $\prod_D \mathbb{Z}/n\mathbb{Z}$, a torsion element, so $\lim \gamma = 0$.

3.3 Cyclic groups in metric topology

The compactification of $\mathbb{Z}$ in the standard metric topology is obviously $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$, with $m + \infty = \infty$ and $\infty + \infty$ equal to any element of $\bar{\mathbb{Z}}$ (the relation). We use the same language as in 3.1 plus the language of semigroups having $x + y = z$ as the basic relation (strictly speaking $\bar{\mathbb{Z}}$ is not a semigroup).

Define a metric on $\mathbb{Z}/n\mathbb{Z}$ as the metric of the regular n-gon with side 1 on the plane.

We identify elements of $\prod_D \mathbb{Z}/n\mathbb{Z}$ with sequences $a = \{a(n) \in \mathbb{Z}/n\mathbb{Z} : n \in \mathbb{N}\}$ modulo D, any given non-principal ultrafilter.

For $m \in \mathbb{Z}$ set $\mathrm{colim}(m)(n) = m + n\mathbb{Z}$, $\mathrm{colim} : \mathbb{Z} \to \prod_D \mathbb{Z}/n\mathbb{Z}$, which is obviously an injective homomorphism into $\prod M_n$. Define

$$\lim a = \begin{cases} m, & \text{if } \{n \in \mathbb{N} : a(n) = m + n\mathbb{Z}\} \in D \\ \infty, & \text{otherwise} \end{cases}$$

In other words, all bounded elements of $\prod_D \mathbb{Z}/n\mathbb{Z}$, which have to be eventually constant, specialise to their eventual value, and the rest go into ∞.

This is a surjective homomorphism onto $\bar{Z}$ in the language of semigroups and the language for metric. Here we also assume that the symbol $\hat{m}$ for each integer m is in the language, and $\hat{m}$ in $\mathbb{Z}/n\mathbb{Z}$ defines the residue class of m modulo n. Note that positive $\exists$-formulas are preserved by a homomorphism so we may assume that the sets defined by such formulas are also in our class $\mathcal{C}$ of closed sets.

Proposition 2 *$\bar{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ is strongly approximated by $\mathbb{Z}/n\mathbb{Z}$ in the topology based on the closed sets given by the positive $\exists$-formulas.*

The two approximations of cyclic groups discussed above are clearly different since the above element γ converges to ∞ in the metric approximation.

3.4 q-approximation of cyclic groups.

Consider a cyclic multiplicative subgroup $\Gamma_q := q^{\mathbb{Z}}$ of $\mathbb{C}^*$ with the generator $q = e^{2\pi i h}$, $h \in \mathbb{R}$. We compactify Γ_q

$$\bar{\Gamma}_q = \begin{cases} \Gamma_q \cup \{0, \infty\}, & \text{if } q \text{ is root of unity} \\ \Gamma_q \cup \{0, \infty\}, & \text{otherwise} \end{cases}$$

with the interpretation of the multiplication relation on the extra elements:

- $x \cdot 0 = 0$, for all $x \neq \infty$;
- $x \cdot \infty = \infty$, for all $x \neq 0$;
- $\infty \cdot 0 = x$, for any x.

We define the distance predicates only to hold between 1 and q, so

$$d_r^{\leq}(1, q) \text{ iff } h \leq r, \quad d_r^{\geq}(1, q) \text{ iff } h \geq r$$

and

$$\neg d_r^{\leq}(q^n, q^m) \text{ for all } r \in \mathbb{R}, n \neq 0, m \neq 1.$$

In other words we are only interested in the size of the generator, and assume it equal to h.

The language $\mathcal{C}$ includes:

- the ternary relation $x \cdot y = z$;
- the symbol q for the generator of $q^{\mathbb{Z}}$;
- the distance predicates $d_r(x, y)$ interpreted as above;
- all relations defined from the above by positive $\exists$-formulas.

Note that among the latter relations there are unary predicates, for all $n > 0$,

$$P_n(x) \equiv \exists y\, y^n = x.$$

Note that, for $n > 1$, $\neg P_n(q)$ holds.

The corresponding predicates are also present in the language of $\bar{\mathbb{Z}}$, in the additive form $\exists y\, y + \ldots + y = x$ (n-multiple sum), but these predicates are trivial in $\bar{\mathbb{Z}}$, since every $x \in \bar{\mathbb{Z}}$ satisfies $\infty + \ldots + \infty = x$.

Now we investigate when a sequence of (compactified) $\bar{\Gamma}_{q_i}$, $i \in \mathbb{N}$, approximates the compactification of a $\bar{\Gamma}_q$ along an ultrafilter D. That is when

$$\lim_D \bar{\Gamma}_{q_i} = \bar{\Gamma}_q. \tag{1}$$

We distinguish two cases.

Case A. Neither of the q_i is a root of unity.

Case B. All the q_i are roots of unity.

Note that using the fact that D is an ultrafilter we may assume without loss of generality that these are the only alternatives.

3.5 Proposition. *(i) Suppose q is not a root of unity.*

In case A, (1) holds if and only if

(a) $\lim_D q_i = q$ in the metric on the unit circle.

In case B, we may assume

$$q_i = \exp 2\pi i \frac{M_i}{N_i}, \quad (M_i, N_i) = 1, \quad 0 < M_i < N_i \text{ integers}$$

Then (1) holds if and only if conditions (a) is satisfied along with the condition

(b) for any natural number m,

$$\{i \in \mathbb{N} : m | N_i\} \in D.$$

(ii) For $q = 1$, the condition (1) holds if and only if $\lim_D q_i = 1$ in the metric on the unit circle.

Proof (i) Case A. Assuming (1) we will have for the limit point $\tilde{q} = \lim_D q_i$, $\mathrm{dist}(\tilde{q}, 1) = r$, for $r = \lim_D \mathrm{dist}(q_i, 1)$. Since, by definition, distance predicates are preserved under lim, $\mathrm{dist}(q, 1)$, so (a) holds.

Conversely, if (a) holds, then in the ultraproduct the element γ corresponding to the generator (that is $\gamma(i) = q_i$ for all $i \in X \in D$) is of infinite order and, since each q_i generates an infinite cyclic group, we can identify the ultraproduct of the compactified groups with $\gamma^{{}^*\mathbb{Z}} \cup \{0, \infty\}$. We consider here ${}^*\mathbb{Z}$ as an ordered additive group. Define

$$\lim \gamma^\eta = \begin{cases} \infty \text{ if } \eta > \mathbb{Z} \\ 0 \quad \text{if } \eta < \mathbb{Z} \\ q^n \text{ if } \eta = n \in \mathbb{Z} \end{cases}$$

This clearly is a homomorphism onto $q^{\mathbb{Z}} \cup \{0, \infty\} = \bar{\Gamma}_q$ with respect to all the relations in the language. Observe, that in a special case we may take $\gamma = q$.

Now we consider the case B. As above, condition (a) is necessary for (1) to hold.

Now we claim that (b) is also necessary. Indeed, assuming that m does not divide N_i along the ultrafilter, let $m = m_1 m_2$ such that $m_1 | N_i$ and $(m_2, N_i) = 1$ for all $i \in X$, some $X \in D$, $m_2 \neq 1$. We may assume $m = m_2$. For all $i \in X$, let u_i, v_i be the integers such that $u_i m + v_i N_i = 1$. Correspondingly,

$$u_i m \equiv 1 \bmod N_i.$$

It follows that

$$(q_i^{u_i})^m = q_i,$$

that is $P_m(\gamma)$ holds, in contrast with $\neg P_m(q)$. So there is no homomorphism taking γ to q.

It remains to prove that, assuming (a) and (b), there is a homomorphism lim satisfying (1).

Note that (a) also implies that γ is of infinite order. Now condition (b) tells us that the ultraproduct of groups $q_i^{\mathbb{Z}}$ is isomorphic to the ultraproduct $\prod_D \mathbb{Z}/N\mathbb{Z}$ along a profinite ultrafilter, as in 3.2. By 3.2, factoring by the torsion subgroup we get a surjective group homomorphism

$$\phi : \prod_D q_i^{\mathbb{Z}} \to q^{{}^*\mathbb{Z}}.$$

This can be trivially extended to include 0 and ∞. Now we use the surjective homomorphism

$$\lim : q^{{}^*\mathbb{Z}} \to q^{\mathbb{Z}} \cup \{0, \infty\}$$

constructed above and finally the composition $\lim \circ \phi$ is the limit map that proves that the sequence $q_i^{\mathbb{Z}}$ approximates $q^{\mathbb{Z}} \cup \{0, \infty\}$.

4 Noncommutative example

4.1 Let $S_q = S_q(\mathbb{C})$ be the structure on $\bar{\mathbb{C}} = \mathbf{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ with the standard Zariski language extended to include the predicate $\bar{\Gamma}$ and the constant symbol q, interpreted as $\bar{\Gamma}_q$ and q, correspondingly.

We need to clarify the interpretation of multiplication on $\bar{\mathbb{C}}$. We consider the graph $P \subseteq \mathbb{C}^3$ of multiplication on $\mathbb{C}$ ($P(x, y, z) \equiv x \cdot y = z$) and extend it to $\bar{P} \subseteq (\bar{\mathbb{C}})^3$, its Zariski (and metric) closure. Clearly, $\bar{P}$ is not the graph of a function anymore, namely

$$\vDash \bar{P}(0, \infty, z) \text{ for every } z \in \bar{\mathbb{C}}.$$

We write this fact equivalently in the form $0 \cdot \infty = \bar{\mathbb{C}}$.

Note, that we also have $\bar{\Gamma} \subset \bar{\mathbb{C}}$ and the multiplication on $\bar{\Gamma}$ (see 3.4 which we consider to be given by the restriction $\bar{P} \cap \bar{\Gamma}^3$. We use correspondingly the notation

$$0 \cdot_{\Gamma} \infty = \bar{\Gamma}, \quad 0 \cdot_{\mathbb{C}} \infty = \bar{\mathbb{C}}$$

to express, when necessary, the fact that multiplication is carried out in terms of the substructure or the whole structure.

Remarks. 1. In this topological language the relation

$$E_q(x_1, x_2) :\equiv \exists \gamma \in \bar{\Gamma}_q x_2 = \gamma x_1$$

is closed and defines on $\mathbb{C}^*$ an equavalence relation (Γ-orbits).

2. In case $q = 1$, the above relation on $\mathbb{C}^*$ is trivial.

Proposition Suppose the assumptions of 3.5 are satisfied, that is

$$\lim_D \bar{\Gamma}_{q_i} = \bar{\Gamma}_q.$$

Then

(i) There exists a specialisation

$$\pi : \prod_i S_{q_i}/D \to S_q$$

realising an approximation of S_q by S_{q_i} along D.

(ii) In particular, S_q is quasi-compact.

(iii) Moreover, in case when Γ_q is infinite, one can choose π so that for every $x \in \prod_i S_{q_i}/D$,

$$\pi(x \cdot \prod_i \Gamma_{q_i}/D) = c\bar{\Gamma}_q, \text{ for some } c \in \mathbb{C}^*.$$

Proof We construct a specialisation π using $\lim_D$. Note that the universe of $\prod_i S_{q_i}/D$ is ${}^*\bar{\mathbb{C}}$. We fix from now on the notation

$${}^*\Gamma := \prod_i \Gamma_{q_i}/D.$$

First set

$$\pi_0(\gamma) = \lim_D(\gamma), \text{ for } \gamma \in {}^*\Gamma,$$
$$\pi_0(u) = u, \text{ for } u \in \bar{\mathbb{C}}$$

We claim that π_0 preserves Zariski predicates on ${}^*\bar{\mathbb{C}}$. Recall the following.

Fact ([8]) Γ satisfies the Lang property and so, for any algebraic variety $V \subseteq \bar{\mathbb{C}}^n$, $\Gamma^n \cap V$ is a finite union of cosets of definable (in the language of groups) subgroups of Γ^n. Moreover, if V is $\mathbb{C}$-definable, then so are the cosets.

It follows that $V({}^*\bar{\mathbb{C}}) \cap {}^*\Gamma$ is preserved by $\lim_D$, so π_0 is a Zariski topology preserving map on ${}^*\Gamma$.

Clearly, π_0 also preserves the predicate Γ. Now extend π_0 to a total specialisation $\pi : {}^*\bar{\mathbb{C}} \to \bar{\mathbb{C}}$, for Zariski (algebraic) predicates. Since π_0 preserves Γ, so does π. This proves (i).

The statement (ii) of the proposition follows when one considers $q_i = q$ for all i, as in 2.2.

(iii) We do this by extending π_0 to π in a more elaborate way, constracting extensions π_α, $\alpha < 2_0^{\aleph}$ (assuming CH). Let π_α be constracted, with domain $A = A_\alpha$, invariant by multiplication by ${}^*\Gamma$ by assumption. We may assume that A is a multiplicative subgroup. We need to extend it to a new element $x \in {}^*\mathbb{C}$.

The case when x is transcendental over A is easy, so we assume x is algebraic over A and satisfies

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0, \quad a_i \in A \tag{2}$$

In case $\pi_\alpha(a_i) \in \mathbb{C}$ for all i, clearly $\pi(x)$ is forced to be finite (that is in $\mathbb{C}$). So we assume that at least for one positive $j \leq n$, $\pi_\alpha(a_j) = \infty$. Among such

a_j we choose one that satisfies the maximality property:

$$\pi_\alpha\left(\frac{a_i^{\frac{1}{i}}}{a_j^{\frac{1}{j}}}\right) \in \mathbb{C}, \text{ for all } i$$

equivalently

$$\pi_\alpha\left(\frac{a_i^{j}}{a_j^{i}}\right) \in \mathbb{C}, \text{ for all } i. \tag{3}$$

Now, by induction hypothesis, there exists $\gamma_0 \in {}^*\Gamma$ such that $\pi_\alpha(a_j\gamma_0) \in \mathbb{C}^*$. For some integer $0 \le k < j$, there is $\gamma_1 \in {}^*\Gamma$ such that $q^k\gamma_0 = \gamma_1^j$. In other words, the order of a_j in the corresponding valuation is equal to that of γ_1^{-j}.

Set $y := x\gamma_1$. Then (2) is equivalent to

$$y^n + a_1\gamma_1 y^{n-1} + \ldots + a_j\gamma_1^j y^j + \ldots + a_n\gamma^n = 0.$$

Note that by our choices, for the new coefficients,

$$\pi_\alpha(a_i\gamma_1^i) \in \mathbb{C} \text{ iff } \pi_\alpha(a_i^j\gamma_1^{ij}) \in \mathbb{C} \text{ iff } \pi_\alpha(a_i^j a_j^{-i}) \in \mathbb{C}$$

and each is true by (3). It follows that we can extend π_α to $\pi_{\alpha+1}$ so that $\pi_{\alpha+1}(y) = c \in \mathbb{C}^*$, so (iii) is satisfied in the end of the inductive process. □

4.2 Noncommutative 2-tori. Consider a $\mathbb{C}$-algebra $\mathcal{A}_q$ generated by "operators" U, U^{-1}, V, V^{-1} satisfying the relation

$$VU = qUV,$$

for some q as above. We continue with the rest of notation as above.

Fix a pair $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$. We will construct two $\mathcal{A}_q$-modules $M_{|u,v>}$ and $M_{<v,u|}$.

The module $M_{|u,v>}$ generated by elements labeled $\{\mathbf{u}(\gamma u, v) \ : \ \gamma \in \Gamma\}$ satisfying

$$\begin{array}{l} U : \mathbf{u}(\gamma u, v) \mapsto \gamma u\, \mathbf{u}(\gamma u, v) \\ V : \mathbf{u}(\gamma u, v) \mapsto v\, \mathbf{u}(q^{-1}\gamma u, v) \end{array} \tag{4}$$

Define also the module $M_{<v,u|}$ generated by $\{\mathbf{v}(\gamma v, u) \ : \ \gamma \in \Gamma\}$ satisfying

$$\begin{array}{l} U : \mathbf{v}(\gamma v, u) \mapsto u\, \mathbf{v}(q\gamma v, u) \\ V : \mathbf{v}(\gamma v, u) \mapsto \gamma v\, \mathbf{v}(\gamma v, u) \end{array} \tag{5}$$

Eventually we would like to see the both modules as submodules of an ambient module, which in case q is a root of unity will coincide with each of the modules, that is $M_{|u,v>} = M_{<v,u|}$ in this case.

Now let $\phi : \mathbb{C}^*/\Gamma \to \mathbb{C}^*$ be a (non-definable) "choice function" which chooses an element in every class $u\Gamma$ and $\Phi = \mathrm{range}\,\phi$, a set of representatives of $\mathbb{C}^*/\Gamma$. We will work with $\Phi^2 = \Phi \times \Phi$, a set of representatives of $\mathbb{C}^*/\Gamma \times \mathbb{C}^*/\Gamma$.

Let

$$\mathbf{U}_\phi := \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \langle u, v\rangle \in \Phi^2, \gamma_1, \gamma_2 \in \Gamma\}$$

a subset of

$$\bigcup_{u,v} M_{|u,v>}$$

and

$$\mathbf{V}_\phi := \{\gamma_1 \cdot \mathbf{v}(\gamma_2 v, u) : \langle u, v\rangle \in \Phi^2, \gamma_1, \gamma_2 \in \Gamma\},$$

a subset of

$$\bigcup_{v,u} M_{<v,u|}$$

Our language for the structure will have unary predicates $\mathbf{U}$ and $\mathbf{V}$ for the sorts $\mathbf{U}_\phi$ and $\mathbf{V}_\phi$.

We will also consider the definable sets

$$\mathbb{C}^*\mathbf{U}_\phi := \{x \cdot \mathbf{u}(\gamma u, v) : \langle u, v\rangle \in \Phi^2, x \in \mathbb{C}^*, \gamma \in \Gamma\}$$

$$\mathbb{C}^*\mathbf{V}_\phi := \{y \cdot \mathbf{v}(\gamma v, u) : \langle u, v\rangle \in \Phi^2, y \in \mathbb{C}^*, \gamma \in \Gamma\},$$

which can be defined as sets of pairs factored by an equivalence relation E that identifies $\gamma \in \Gamma$ as an element of $\mathbb{C}^*$:

$$\mathbb{C}^*\mathbf{U}_\phi = (\mathbb{C} \times \mathbf{U}_\phi)/E, \quad \mathbb{C}^*\mathbf{V}_\phi = (\mathbb{C} \times \mathbf{V}_\phi)/E.$$

We also consider the **pairing**

$$\langle \cdot \mid \cdot \rangle : \ \mathbf{V}_\phi \times \mathbf{U}_\phi \to \Gamma$$

(as a ternary relation on $\mathbf{V} \times \mathbf{U} \times \mathbb{C}$) defined as follows. For $q^s\,\mathbf{v}(q^m v, u) \in \mathbf{V}_\phi$ and $q^r\,\mathbf{u}(q^k u, v) \in \mathbf{U}_\phi$ set

$$\langle q^s\,\mathbf{v}(q^m v, u) | q^r\,\mathbf{u}(q^k u, v)\rangle = q^{r-s-km} \tag{6}$$

And let $\langle q^s\, \mathbf{v}(v', u) | q^r\, \mathbf{u}(u', v)\rangle$ not defined (the ternary relation does not hold) if $v' \notin \Gamma \cdot v \ \vee \ u' \notin \Gamma \cdot u$.

We also agree to consider the pairing on $\mathbf{U}_\phi \times \mathbf{V}_\phi$ setting

$$\langle q^r\, \mathbf{u}(q^k u, v) | q^s\, \mathbf{v}(q^m v, u)\rangle = q^{km+s-r} = \langle q^s\, \mathbf{v}(q^m v, u) | q^r\, \mathbf{u}(q^k u, v)\rangle^{-1}$$

which corresponds to taking the complex conjugate of (6).

Remark 1. The definable relations on $\mathbf{U}$ (and an analogous on $\mathbf{V}$)

$$\exists \gamma_1, \gamma_2 \in \Gamma \exists b \in \mathbf{V}\ (\langle b|a_1\rangle = \gamma_1\ \&\ \langle b|a_2\rangle = \gamma_2)$$

is an equivalence relation. An equivalence class defined by this relation has the form

$$\{\gamma' \mathbf{u}(\gamma'' u, v) :\ \gamma', \gamma'' \in \Gamma\}$$

for some $u, v \in \Phi$. Consequently, the pairing uniquely determines the modules $M_{|u,v>}$ and $M_{<v,u|}$ once any of $\mathbf{u}(\gamma u, v)$ or $\mathbf{v}(\gamma v, u)$ is known.

We will call the 3-sorted structure $(\mathbf{U}_\phi, \mathbf{V}_\phi, \mathbb{C}^*)$ with the action of U and V satisfying (4), (5) and a pairing a **(complexified) quantum 2-torus** $T_q^2(\mathbb{C})$.

Remark 2. For q root of unity $T_q^2(\mathbb{C})$ was, in fact, constructed in [6], in a different language.

Remark 3. It is possible and crucial for applications to consider extensions of $\mathcal{A}_q$ by other operators and corresponding expansion of this structure by other sorts, that can be interpreted as eigenvectors of the new operators, with the pairing extended to the new sorts. We do some of this in [7].

Remark 4. It is useful to also use another equivalent system of notation and write symbolically, for a vector $\mathbf{a}$:

$$\mathbf{a} = \sum_{\gamma \in \Gamma} c_\gamma \mathbf{u}(\gamma u, v),$$

if

$$\langle \mathbf{a} | \mathbf{u}(\gamma u, v)\rangle = c_\gamma$$

and the pairing with all other **u**-elements is 0. Analogously, with respect to **v**-elements.

In particular,

$$\mathbf{v}(q^m v, u) = \sum_m q^{km}\mathbf{u}(q^k u, v),$$

$$\mathbf{u}(q^k u, v) = \sum_p q^{-kp}\mathbf{v}(q^p v, u).$$

4.3 The $*$-conditions and the case $q = 1$. In noncommutaive geometry an important role is played by the extra $*$-conditions. In case of the non-commutative tori it is the assumption that U and V must be considered as unitary operators. This will have the consequence that the eigenvalues of U and V are on the unit circle $\mathcal{S}$ (complex numbers of modulus 1) and so we will be forced to consider just the substructure of $(\mathbf{U}_\phi, \mathbf{V}_\phi, \mathbb{C})$ with $u, v \in \mathcal{S}$.

In case $q = 1$ (without $*$-assumptions) $\Phi = \mathbb{C}^*$ and every element of $\mathbf{U}_\phi$ as well as $\mathbf{V}_\phi$ is determined just by $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$. So, in fact, each of these sorts can be simply identified with the algebraic 2-torus $\mathbb{C}^* \times \mathbb{C}^*$ (we can say that the bundles are trivial) and the actions of U and V are definable in the usual algebraic (Zariski) structure on $\mathbb{C}^*$.

Under the assumption that U and V are unitary we will get the corresponding classical structure on $\mathcal{S}^2$.

4.4 Proposition. *Given $q \in \mathbb{C}$ any two structures of the form $T_q^2(\mathbb{C})$ are isomorphic over $\mathbb{C}$. In other words, the isomorphism type of $T_q^2(\mathbb{C})$ does not depend on the system of representatives Φ.*

Proof Let $\pi : (\mathbf{U}, \mathbf{V})_\phi \to (\mathbf{U}, \mathbf{V})_\psi$ be a partial elementary monomorphism such that if $\langle u, v\rangle \in D = \operatorname{dom}\pi$ then $\langle uq^k, vq^m\rangle \in \operatorname{dom}\pi$, for all $k, m \in \mathbb{Z}$. We want to extend π to a new coset $u_e q^{\mathbb{Z}} \times v_e q^{\mathbb{Z}}$.

We may assume that $\langle u_e, v_e\rangle \in \Phi^2$. Let $\langle u_0, v_0\rangle \in \mathrm{F}^2$ satisfy

$$\langle u_e, v_e\rangle^\frown D \equiv \langle u_0, v_0\rangle^\frown \pi(D).$$

Consider the bases $\{\mathbf{u}(q^k u_e, v_e) \ : \ k \in \mathbb{Z}\}$ and $\{\mathbf{v}(q^k v_e, u) \ : \ k \in \mathbb{Z}\}$ satisfying (4),(5) and (6). Let $\langle u_g, v_g\rangle$ the Φ point of the coset $u_0 q^{\mathbb{Z}} \times v_0 q^{\mathbb{Z}}$.

We want to find a subset of $\mathbf{U}_\psi$, $\{q^{n(k)}.\mathbf{u}(q^k u_g, v_g) : \ k \in \mathbb{Z}\}$, and a subset of $\mathbf{V}_\psi$, $\{q^{m(k)}.\mathbf{v}(q^k v_g, u_g) \ : \ k \in \mathbb{Z}\}$, satisfying (4), (5) and (6) for u_0, v_0 when we set

$$\mathbf{u}(q^k u_0, v_0) := q^{n(k)}.\mathbf{u}(q^k u_g, v_g), \quad \mathbf{v}(q^k v_0, u_0) := q^{m(k)}.\mathbf{v}(q^k v_g, u_g), \text{ all } k \in \mathbb{Z}.$$

By assumptions $u_0 = q^s u_g$, $v_0 = q^t v_g$, for some $s, t \in \mathbb{Z}$. Set

$$\mathbf{u}(u_0, v_0) := q^{st}\mathbf{u}(q^s u_g, v_g) \text{ and } \mathbf{u}(q^k u_0, v_0) := v_0^k V^{-k}\mathbf{u}(u_0, v_0), \text{ for } k \in \mathbb{Z}.$$

This satisfies (4) by construction. Moreover,

$$\mathbf{u}(q^k u_0, v_0) = q^{kt+st}.\mathbf{u}(q^{k+s}u_g, v_g) \tag{7}$$

Define

$$\mathbf{v}(v_0, u_0) := \mathbf{v}(q^t v_g, u_g) \quad \text{ and } \mathbf{v}(q^k v_0, u_0) := u_0^{-k} U^k \mathbf{v}(v_0, u_0), \text{ for } k \in \mathbb{Z},$$

which satisfies (5), and we also have

$$\mathbf{v}(q^m v_0, u_0) = q^{-sm} \cdot \mathbf{v}(q^{m+t}v_g, u_g) \tag{8}$$

One can now see that by definition

$$\langle \mathbf{v}(u_0, v_0) | \mathbf{u}(v_0, u_0) \rangle = \langle \mathbf{v}(q^t v_g, u_g) | q^{st} \cdot \mathbf{u}(q^s u_g, v_g) \rangle = 1,$$

$$\begin{array}{l} \langle \mathbf{v}(q^m v_0, u_0) | \mathbf{u}(q^k u_0, v_0) \rangle = \langle q^{-sm} \cdot \mathbf{v}(q^{m+t}v_g, u_g) \mid q^{st+kt} \cdot \mathbf{u}(q^s u_g, v_g) \rangle = \\ = q^{-sm-st-kt}.\langle \mathbf{v}(q^{m+t}v_g, u_g) \mid \mathbf{u}(q^{k+s}u_g, v_g) \rangle = q^{km} \end{array} .$$

So, the two (U, V)-systems are isomorphic.

4.5 Compactification of T_q^2. We introduce a topological structure $\bar{T}_q^2(\mathbb{C})$ such that $T_q^2(\mathbb{C})$ is open in the former. This is a 3-sorted structure $(\bar{\mathbf{U}}_\phi, \bar{\mathbf{V}}_\phi, \bar{\mathbb{C}})$, with $\bar{\mathbb{C}}$ as in 4.1.

We extend the sort $\mathbf{U}$ to $\bar{\mathbf{U}}$.

Set

$$\bar{\mathbf{U}}_\phi := \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \langle u, v \rangle \in \Phi^2, \gamma_1, \gamma_2 \in \bar{\Gamma}\}$$

and

$$\bar{\mathbf{V}}_\phi := \{\gamma_1 \cdot \mathbf{v}(\gamma_2 v, u) : \langle u, v \rangle \in \Phi^2, \gamma_1, \gamma_2 \in \bar{\Gamma}\}.$$

These is extended naturally to $\bar{\mathbb{C}}\bar{\mathbf{U}}_\phi$ and $\bar{\mathbb{C}}\bar{\mathbf{V}}_\phi$ as before.

With the new elements the action is defined as the binary relation $\bar{\mathbb{C}}\bar{\mathbf{U}} \to \bar{\mathbb{C}}\bar{\mathbf{U}}$, so multivalued at some points.

- $U : \; x \cdot \mathbf{u}(0 \cdot u, v) \mapsto 0 \cdot_{\mathbb{C}} x \cdot \mathbf{u}(0 \cdot u, v)$,
- $U : \; x \cdot \mathbf{u}(\infty \cdot u, v) \mapsto \infty \cdot_{\mathbb{C}} x \cdot \mathbf{u}(\infty \cdot u, v)$,

- $U^{-1} : x \cdot \mathbf{u}(0 \cdot u, v) \mapsto \infty \cdot_{\mathbb{C}} x \cdot \mathbf{u}(0 \cdot u, v)$,
- $U^{-1} : x \cdot \mathbf{u}(\infty \cdot u, v) \mapsto 0 \cdot_{\mathbb{C}} x \cdot \mathbf{u}(\infty \cdot u, v)$,
- $V : \ x \cdot \mathbf{u}(0 \cdot u, v) \mapsto v\bar{\Gamma} \cdot \mathbf{u}(0 \cdot u, v)$,
- $V : \ \gamma \cdot e(\infty \cdot u, v) \mapsto v\bar{\Gamma} \cdot \mathbf{u}(\infty \cdot u, v)$.

Similarly the action of U and V on $\bar{\mathbf{V}}$.

We also extend the pairing to the extra elements

$$\begin{array}{ll} \langle \gamma' \mathbf{v}(0 \cdot v, u) | \gamma'' \mathbf{u}(\gamma u, v) \rangle = \bar{\Gamma}, & \langle \gamma' \mathbf{v}(\infty \cdot v, u) | \gamma'' \mathbf{u}(\gamma u, v) \rangle = \bar{\Gamma} \\ \langle \gamma' \mathbf{v}(\gamma v, u) | \gamma'' \mathbf{u}(0 \cdot u, v) \rangle = \bar{\Gamma}, & \langle \gamma' \mathbf{v}(\gamma v, u) | \gamma'' \mathbf{u}(\infty \cdot u, v) \rangle = \bar{\Gamma} \end{array}$$

for every $\gamma, \gamma', \gamma'' \in \bar{\Gamma}$.

This completes the description of $\bar{T}_q^2(\mathbb{C})$.

Remark The structure $\bar{T}_q^2(\mathbb{C})$ is definable in the structure $T_q^2(\mathbb{C})$.

4.6 Topology on $\bar{T}_q^2$. We assume closed:

- any point in sorts $\bar{\mathbf{U}}$, $\bar{\mathbf{V}}$ and $\bar{\mathbb{C}}$;
- all Zariski closed subsets of $\bar{\mathbb{C}}^n$;
- $\bar{\mathbf{U}}$, $\bar{\mathbf{V}}$, $\bar{\mathbb{C}}\bar{\mathbf{U}}_\phi$ and $\bar{\mathbb{C}}\bar{\mathbf{V}}_\phi$;
- the graphs of the actions of $\bar{\mathbb{C}}$ on $\bar{\mathbb{C}}\bar{\mathbf{U}}_\phi$ and $\bar{\mathbb{C}}\bar{\mathbf{V}}_\phi$;
- the graphs of the actions of U and V on $\bar{\mathbb{C}}\bar{\mathbf{U}}_\phi$ and $\bar{\mathbb{C}}\bar{\mathbf{V}}_\phi$;
- the subset $\bar{\Gamma}$ of $\bar{\mathbb{C}}$;
- the graph of the pairing $\langle \cdot \mid \cdot \rangle$;
- all sets positive type-definable from the above.

4.7 Theorem *Suppose the assumptions of 3.5 are satisfied, that is*

$$\lim_D \bar{\Gamma}_{q_i} = \bar{\Gamma}_q.$$

Then there exists a specialisation

$$\sigma : \prod_i \bar{T}_{q_i}^2 / D \to \bar{T}_q^2$$

realising an approximation of $\bar{T}^2_q$ *by* $\bar{T}^2_{q_i}$ *along* D.

In particular, $\bar{T}^2_q$ *is quasi-compact.*

For $q = 1$ $\bar{T}^2_{q_i}$ *converge to the classical* 2*-torus compactified by* 0 *and* ∞ *and the* $*$*-version of* $T^2_{q_i}$ *converge to* $\mathcal{S}^2$.

Proof We use the Proposition of 4.1 and the specialisation π constructed therein.

We define $\sigma(z) := \pi(z)$, for $z \in {}^*\bar{\mathbb{C}}$ (so in sort $\bar{\mathbb{C}}$). This preserves Zariski topology on $\bar{\mathbb{C}}$ and preserves $\bar{\Gamma}$.

For each Γ-coset $\pi(u^*\Gamma)$ we can choose, by 4.1(iii) a representative c from $\mathbb{C}^*$, denote this representative $\psi(u)$. In case $\pi(u)$ is already in $\mathbb{C}^*$ we set $\psi(u) = \pi(u)$. Note that by our definition $\pi(u)\psi(u)^{-1} \in \bar{\Gamma}$.

Now we want to define σ on $\bar{\mathbf{U}}_\phi$ and $\bar{\mathbf{V}}_\phi$.

First we introduce, for $v \in {}^*\bar{\mathbb{C}}$ and $\gamma \in {}^*\bar{\Gamma}$, functions $t(v)$ and $a(\gamma)$,

$$t(v) = \begin{cases} 0 \quad \text{if } \pi(v) \in \mathbb{C}^*, \\ +\infty \text{ if } \pi(v) = 0, \\ -\infty \text{ if } \pi(v) = \infty. \end{cases}$$

$$a(\gamma) = \begin{cases} n \text{ if } \pi(\gamma) = \gamma^n \in \Gamma, \\ +\infty \text{ if } \pi(\gamma) = \infty, \\ -\infty \text{ if } \pi(\gamma) = 0. \end{cases}$$

We are going to use the expression $a(\gamma) \cdot t(v)$, which we define to be 0 if one of the factors is 0, and otherwise calculate it by natural rule as a product of ∞ with a finite number or ∞ taking the signs into account. So, $a(\gamma) \cdot t(v)$ is always 0, $+\infty$ or $-\infty$. Now we interpret the expression

$$q^{a(\gamma)\cdot t(v)} = \begin{cases} 1 \quad \text{if } a(\gamma) \cdot t(v) = 0, \\ \infty \text{ if } a(\gamma) \cdot t(v) = +\infty, \\ 0 \ \text{ if } a(\gamma) \cdot t(v) = -\infty. \end{cases}$$

Denote also $\delta(\gamma, u) := \pi(\gamma u)\psi(u)^{-1}$. Note that for $u \in {}^*\Phi$ and $\gamma \in {}^*\Gamma$, we get $\delta(\gamma, u) \in \bar{\Gamma}$.

We set for $x \in {}^*\bar{\mathbb{C}}$, $\gamma \in {}^*\bar{\Gamma}$ and $u, v \in {}^*\Phi$:

$$\sigma : \begin{cases} x \cdot \mathbf{u}(\gamma \cdot u, v) \mapsto \pi(x) \cdot_{\mathbb{C}} q^{a(\gamma)\cdot t(v)} \cdot \mathbf{u}(\delta(\gamma, u) \cdot \psi(u), \psi(v)) \\ x \cdot \mathbf{v}(\gamma \cdot v, u) \mapsto \pi(x) \cdot_{\mathbb{C}} q^{a(\gamma)\cdot t(u)} \cdot \mathbf{v}(\delta(\gamma, v) \cdot \psi(v), \psi(u)) \end{cases}$$

So, σ preserves $\bar{\mathbf{U}}$, $\bar{\mathbf{V}}$ and $\bar{\mathbb{C}}\bar{\mathbf{U}}$, $\bar{\mathbb{C}}\bar{\mathbf{V}}$.

Also, clearly

$$\sigma : \begin{cases} U\mathbf{u}(\gamma u, v) \mapsto \sigma[U\mathbf{u}(\gamma u), v)] \\ V\mathbf{v}(\gamma v, u) \mapsto \sigma[V\mathbf{v}(\gamma v, u)] \end{cases}$$

that is preserves the action of U on $\bar{\mathbb{C}}\bar{\mathbf{U}}$ and the action of V on $\bar{\mathbb{C}}\bar{\mathbf{V}}$.

Claim σ respects the action of V on $\bar{\mathbb{C}}\bar{\mathbf{U}}$ as well as the action of U on $\bar{\mathbb{C}}\bar{\mathbf{V}}$.

Proof We check it for V. We have

$$x \cdot \mathbf{u}(\gamma \cdot u, v) \mapsto^{\sigma} \pi(x) \cdot_{\mathbb{C}} q^{a(\gamma)\cdot t(v)} \cdot \mathbf{u}(\delta(\gamma, u) \cdot \psi(u), \psi(v)) \mapsto^{V}$$

$$\mapsto^{V} \pi(v) \cdot_{\mathbb{C}} \pi(x) \cdot_{\mathbb{C}} q^{a(\gamma)\cdot t(v)} \cdot \mathbf{u}(q^{-1}\delta(\gamma, u) \cdot \psi(u), \psi(v))$$

and

$$x \cdot \mathbf{u}(\gamma \cdot u, v) \mapsto^{V} v \cdot_{\mathbb{C}} x \cdot \mathbf{u}(q^{-1}\gamma \cdot u, v) \mapsto^{\sigma} \pi(xv) \cdot_{\mathbb{C}} q^{a(q^{-1}\gamma)\cdot t(v)} \mathbf{u}(\delta(q^{-1}\gamma, u)\psi(u), \psi(v)) =$$

$$= \pi(x) \cdot_{\mathbb{C}} \pi(v) \cdot_{\mathbb{C}} q^{a(q^{-1}\gamma)\cdot t(v)} \mathbf{u}(q^{-1}\delta(\gamma, u)\psi(u), \psi(v)).$$

Recall that V is in general a multivalued operation (relation) and so is the multiplication on $\bar{\mathbb{C}}$ and $\bar{\Gamma}$. So it is enough to check that

$$\pi(v) \cdot q^{a(\gamma)\cdot t(v)} = \bar{\mathbb{C}} \tag{9}$$

or

$$\pi(v) \cdot q^{a(\gamma)\cdot t(v)} = \pi(v) \cdot q^{a(q^{-1}\gamma)\cdot t(v)}, \text{ both sides singletons.} \tag{10}$$

The latter happens if $t(v) = 0$, so we assume this is not the case. Then

$$t(v) = +\infty \text{ or } t(v) = -\infty, \tag{11}$$

respectively $\pi(v) = 0$ or $\pi(v) = \infty$. It is also clear that under the assumptions (10) holds unless

$$a(\gamma) = 1 \text{ or } a(\gamma) = 0. \tag{12}$$

So we may assume (11) and (12), which splits into four cases:

(i) $t(v) = +\infty$, $\pi(v) = 0$, $a(\gamma) = 1$

(ii) $t(v) = -\infty$, $\pi(v) = \infty$, $a(\gamma) = 1$

(iii) $t(v) = +\infty$, $\pi(v) = 0$, $a(\gamma) = 0$

(iv) $t(v) = -\infty$, $\pi(v) = \infty$, $a(\gamma) = 1$

In case (i) $q^{a(\gamma)\cdot t(v)} = q^{+\infty} = \infty$ and we have (9).

In case (ii) $q^{a(\gamma)\cdot t(v)} = q^{-\infty} = 0$ and we have again (9).

In case (iii) $q^{a(\gamma)\cdot t(v)} = q^0 = 1$ and $\pi(v) \cdot q^{a(\gamma)\cdot t(v)} = 0$. At the same time $a(q^{-1}\gamma) = -1$, $q^{a(q^{-1}\gamma)\cdot t(v)} = q^{-\infty} = 0$, so $\pi(v) \cdot q^{a(q^{-1}\gamma)\cdot t(v)} = 0$, which gives (10).

In case (iv), similarly, $\pi(v) \cdot q^{a(\gamma)\cdot t(v)} = \infty = \pi(v) \cdot q^{a(q^{-1}\gamma)\cdot t(v)}$.

So σ preserves V. Same argument proves that it preserves U. Claim proved.

Claim σ respects the pairing.

Proof Suppose $\langle \mathbf{v}(\gamma_1 \cdot v, u) \mid \mathbf{u}(\gamma_2 \cdot u, v)\rangle = \gamma$, for $\gamma_1, \gamma_2, \gamma \in {}^*\bar{\Gamma}$, or rather the corresponding ternary relation holds between $\mathbf{v}(\gamma_1 v, u)$, $\mathbf{u}(\gamma_2 u, v)$ and γ. We want to prove that the triple $\sigma\mathbf{v}(\gamma_1 v, u)$, $\sigma\mathbf{u}(\gamma_2 u, v)$ and $\sigma\gamma$ satisfy the same relation.

We may assume that γ_1 and γ_2 are not in $\{0, \infty\}$, since otherwise the statement is trivial. We have then

$$\langle \sigma\mathbf{v}(\gamma_1 \cdot v, u) \mid \sigma\mathbf{u}(\gamma_2 \cdot u, v)\rangle =$$

$$= \langle q^{a(\gamma_1)\cdot t(u)}\mathbf{v}(\delta(\gamma_1, v) \cdot \psi(v), \psi(u)) \mid q^{a(\gamma_2)\cdot t(v)}\mathbf{u}(\delta(\gamma_2, u) \cdot \psi(u), \psi(v))\rangle =$$

$$= q^{a(\gamma_1)\cdot t(u)} \cdot q^{-a(\gamma_2)\cdot t(v)}\langle \mathbf{v}(\delta(\gamma_1, v) \cdot \psi(v), \psi(u)) \mid \mathbf{u}(\delta(\gamma_2, u) \cdot \psi(u), \psi(v))\rangle.$$

We are done if $\delta(\gamma_1, v)$ or $\delta(\gamma_2, u)$ is in $\{0, \infty\}$, so we assume that neither holds. This implies that $\pi(\gamma_1 v), \pi(\gamma_2 u) \in \mathbb{C}^*$.

Furthermore, if both $\pi(v)$ and $\pi(u)$ are in $\mathbb{C}^*$, then so are $\pi(\gamma_1)$ and $\pi(\gamma_2)$, which by definition implies γ_1 and γ_2 are standard, that is of the form q^m, $m \in \mathbb{Z}$. In this case also $\pi(v) = \psi(v)$, $\pi(u) = \psi(u)$ and the statement of the claim holds.

So we may assume that one of these, say $\pi(u)$ is 0 or ∞. Respectively, we will then have $\pi(\gamma_2)$ is ∞ or 0, $t(u)$ is equal to $+\infty$ or $-\infty$ and $a(\gamma_2)$ is equal to $+\infty$ or $-\infty$.

Consider now the case $\pi(v) \in \{0, \infty\}$, so $t(v) \in \{+\infty, -\infty\}$ and $a(\gamma_1) \in \{+\infty, -\infty\}$. We can see from the above that both $a(\gamma_2) \cdot t(v)$ and $a(\gamma_1) \cdot t(u)$ are equal to $+\infty$, so $\langle \sigma\mathbf{v}(\gamma_1 \cdot v, u) \mid \sigma\mathbf{u}(\gamma_2 \cdot u, v)\rangle = \bar{\Gamma}$ and we are done.

In the case $\pi(v) \in \mathbb{C}^*$, $t(v) = 0$, $\psi(v) = \pi(v)$ and $q^{a(\gamma_2)\cdot t(v)} = 1$. Also, $\pi(\gamma_1) \in \mathbb{C}^*$, $\pi(\gamma_1) = q^m$, $a(\gamma_1) = m \in \mathbb{Z}$. Hence,

$$\langle \sigma\mathbf{v}(\gamma_1 \cdot v, u) \mid \sigma\mathbf{u}(\gamma_2 \cdot u, v)\rangle = q^{m\cdot t(u)}$$

Note that by definition γ_1, γ_2 and γ can be identified in $\prod \Gamma_{q_i}/D$ with q^m, q^κ and $q^{m\kappa}$ correspondigly, for m as above and κ nonstandard. Now one checks that since $\pi(q^{\kappa}u) \in \mathbb{C}^*$, $a(q^{\kappa}) = t(u)$. Also, it follows from definitions, $\pi(q^{\kappa}) = q^{a(q^{\kappa})} = q^{t(u)}$ and $\pi(q^{m\kappa}) = q^{mt(u)}$. That is $\pi(\gamma) = q^{mt(u)}$. This proves the claim.

The main statement of the theorem follows. The $q = 1$ statement follows from 4.3. □

5 Approximation by finite structures

5.1 Approximation by finite fields. According to 2.1 we discuss the approximation of a compactification $\bar{K} = K \cup \{\infty\} = \mathbf{P}^1(K)$, when speaking of an approximation of a field K. The standard topology that we will assume for $\bar{K}$ is the topology *generated by the Zariski topology on* $\bar{K}$, that is the smallest quasi-compact topology $\mathcal{T}$ extending the Zariski topology. Equivalently, by [5], these are the fields K such that for any elementary extension ${}^*K \succ K$ there is a specialisation (place) $\pi : {}^*K \to K$.
Conjecture. For an infinite field, $\bar{K}$ is quasi-compact iff K is algebraically closed or K is isomorphic to one of the known non algebraically closed locally compact fields: $\mathbb{R}$ or finite extension of $\mathbb{Q}_p$ or $\mathrm{F}_p\{t\}$.

5.2 Proposition. (i) Any algebraically closed field K of cardinality continuum with respect to the Zariski language is strongly approximable, but not perfectly approximable, by finite fields.

(ii) The field of reals $\mathbb{R}$ in the field language is not approximable by finite fields.

(iii) No locally compact field, other than algebraically closed, is approximable by finite fields.

Proof (i) $\prod_D M_n = \mathrm{F}$ for M_n finite fields is a pseudofinite field of cardinality continuum. Choose M_n and D so that $\mathrm{char}\, \mathrm{F} = \mathrm{char}\, K$. A pseudofinite field is not algebraically closed, which immediately implies that we don't have a perfect approximation.

We will construct a total surjective specialisation $\pi : \mathrm{F} \to \bar{K} = K \cup \{\infty\}$. Obviously there is a partial specialisation, in fact embedding, of the prime field of charF into K. Suppose we have constructed partial $\pi : \mathrm{F} \to \bar{K}$ with $|\mathrm{dom}\, \pi| < 2^{\omega}$. We want to show that

a) given $a \in F \setminus \operatorname{dom} \pi$ we can extend π to a;

b) given $b \in \mathrm{K}$ we can extend π to some $a \in F$ so that $\pi(a) = b$.

For a) just note the general fact that any partial specialisation from a field into an algebraically closed field can be extended to a total one.

For b) choose $a \in F \setminus \operatorname{acl}(\operatorname{dom} \pi)$. Then defining $\pi(a) := b$ is consistent with π being a specialisation.

(ii) It is known ([10]) that F is a *pseudo-algebraically closed* field, that is any absolutely irreducible variety C over F has an F-point.

Claim. The affine curve C given by the equations

$$x^2 + y^2 + 1 = 0; \quad \frac{1}{x^2} + z^2 + 2 = 0$$

is irreducible over $\mathbb{C}$ and so is absolutely irreducible.

Proof. It is well known that $x^2 + y^2 + a = 0$, for $a \neq 0$, with any of the point removed is biregularly isomorphic to $\mathbb{C}$, and so irreducible. For the same reason the subvariety of $\mathbb{C}^2$ given by $\frac{1}{x^2} + z^2 + 2 = 0$ is also irreducible. We also note that the natural embeddings of both varieties into $\mathbf{P}^2$ are smooth.

The curve C projects into (x, y)-plane as the curve C_{xy} given by $x^2 + y^2 + 1 = 0$ and into the (x, z)-plane as the curve C_{xz} given by $\frac{1}{x^2} + z^2 + 2 = 0$.

Suppose towards a contradiction that $C = C_1 \cup C_2$ with C_1 an irreducible curve, $C_1 \neq C$, and C_2 Zariski closed. We denote $\bar{C}$, $\bar{C}_1$ and $\bar{C}_2$ the corresponding closures in the projective space $\mathbf{P}^3$.

Consider the projection $\operatorname{pr}_{xy} : \bar{C}_1 \to \bar{C}_{xy}$. This is surjective and the order of the projection is either 1 or 2. In the second case $\operatorname{pr}_{xy}^{-1}(a) \cap \bar{C}_1 = \operatorname{pr}_{xy}^{-1}(a) \cap \bar{C}$ for all $a \in \bar{C}_{xy}$, so $C = C_1$ and we are left with the first case only. In this case pr_{xy} is an isomorphism between $\bar{C}_1$ and $\bar{C}_{xy}$. It is also clear in this case that C_2 must be a curve, and pr_{xy} also an isomorphism from $\bar{C}_2$ to $\bar{C}_{xy}$. The points of intersection of C_1 and C_2 are the points over $a \in C_{xy}$ where $|\operatorname{pr}_{xy}^{-1}(a) \cap C| = 1$. One immediately sees that this can only be the points where $z = 0$, $x^2 = -\frac{1}{2}$, $y^2 = -\frac{1}{2}$.

We can apply the same arguments to the projection pr_{xz} onto $\bar{C}_{xz}$ and find that the points of intersection of C_1 and C_2 must satisfy $y = 0$, $x^2 = 1$ and $z^2 = -2$. The contradiction. Claim proved.

Now we prove that the existence of a total specialisation $\pi : \mathrm{F} \to \mathbb{R} \cup \{\infty\}$ or $\pi : \mathrm{F} \to \mathbb{R} \cup \{+\infty, -\infty\}$ leads to a contradiction.

By above there exist a point (x, y, z) in $C(\mathrm{F})$. Then either $\pi(x)$ or $\pi(\frac{1}{x}) \in \mathbb{R}$ (are finite). Let us assume $\pi(x) \in \mathbb{R}$. Then necessarily $\pi(y) \neq \infty$, since $\pi(x)^2 + \pi(y)^2 + 1 = 0$, but the latter contradicts that $x^2 + y^2 \geq 0$ in $\mathbb{R}$.

(iii) If L is a residue field for a valued field K, then the residue map $\bar{K} \to \bar{L}$ is a place. So assuming there is a sujective place $\bar{\mathrm{F}} \to \bar{K}$ we get a suregctive place $\bar{\mathrm{F}} \to \bar{L}$. This is not possible for a PAC-field, by [10], Corollary 11.5.5. □

5.3 The following, we believe, is crucial for discrete approximation for gauge field theories.

Problem

1. Is the group SO(3) approximable by finite groups in the group language?

2. More generally, let G be a compact simple Lie group. Is G approximable by finite groups in the group language? Equivalently (assuming for simplicity the continuum hypothesis), is there a sequence of finite groups G_n, $n \in \mathbb{N}$, an ultrafilter D on $\mathbb{N}$ and a surjective group homomorphism

$$\prod_n G_n/D \to G.$$

Remark This problem has an easy solution (in fact well-known to physicists) if we are content with G_n to be quasi-groups, that is omit the requirement of associativity of the group operation:

For each n choose an $\frac{1}{n}$-dense finite subset $G(n) \subset G$ of points. For $a, b \in G(n)$ set $a * b$ to be a point in $G(n)$ which is at a distance less than $\frac{1}{n}$ from the actual product $a \cdot b$ in G. Now set, for $\gamma \in \prod_n G_n$,

$$\lim \gamma = g \text{ iff } \{n \in \mathbb{N} : \ \mathrm{dist}(\gamma(n), \, g) \leq \frac{1}{n}\} \in D,$$

which is in fact the standard part map. Then clearly

$$\lim(\gamma_1 * \gamma_2) = \lim \gamma_1 \cdot \lim \gamma_2,$$

that is the map is a homomorphism.

References

[1] B. Zilber, *Zariski Geometries*, forthcoming book, Cambridge University Press.

[2] B. ZIlber, *A Class of Quantum Zariski Geometries*, in Model Theory with Applications to Algebra and Analysis, I, Volume 349 of LMS Lecture Notes Series, Cambridge University Press, 2008.

[3] E.Hrushovski and B.Zilber, *Zariski Geometries*. Journal of AMS, 9(1996), 1-56

[4] J.M.Rabin, *Introduction to quantum field theory* in **Geometry and quantum field theory**, K.Uhlenbeck, D.Freed, editors Providence, R.I : American Mathematical Society, 1995, pp.185-270

[5] Weglorz, B. Equationally compact algebras. I. Fund.

[6] B.Zilber, *A class of quantum Zariski geometries*

[7] B.Zilber, *The noncommutative torus and rigorous Dirac calculus,* webpage

[8] O.Belegradek and B.Zilber, *The model theory of the field of reals with a subgroup of the unit circle* J. London Math. Soc. (2) 78 (2008) 563-579

[9] M.Rieffel, *Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance*. Mem. Amer. Math. Soc. 168 (2004) no. 796, 67-91; math.OA/0108005

[10] M.D. Fried and M. Jarden **Field arithmetic**, Second edition, revised and enlarged by Moshe Jarden, Ergebnisse der Mathematik (3) 11, Springer, Heidelberg,2005.