# On model theory, non-commutative geometry and physics


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## Plan

I. Generalities on physics, logical hierarchy of structures and Zariski geometries.

II. Structural approximation.

III. Zariski geometries as noncommutative spaces.

IV. Crush course in physics.

V. Zariski geometries for quantum Hamiltonian systems.

VI. Time evolution and Feynman propagator for some Hamiltonians.

First question: in what sense the (mathematical) physics we study reflects the "real universe"? If our mathematical models are approximations to reality, then in what sense approximation?

A plausible answer to the first part of the question: *the real universe is a structure, say* $\mathbf{M}$, *that is a domain (set of points, "events", "particles",...) with some relations* $R(x_1, ..., x_n)$ *between its elements.*
*We want to distinguish relations defining (topologically)* **closed** *sets.*

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*We want to distinguish relations defining (topologically)* **closed** *sets.*

Note, **this is not the geometer's point of view.**

What is an *idealised model* of such a real universe. Another structure $\tilde{\mathbf{M}}$ such that

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(ii) close points in $\mathbf{M}$ may be indistinguishable in $\tilde{\mathbf{M}}$.

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A possible formalisation of this: $\mathbf{M} \in \{\mathbf{M}_i : i \in I\}$ and for some ultrafilter $U$ on $I$ there is $\lim_U$, a surjective homomorphism (preserves all primitive relations)

$$\lim_U : \prod_i \mathbf{M}_i / U \to \tilde{\mathbf{M}}.$$

We say in this case that the sequence of structures $\mathbf{M}_i$ approximates $\tilde{\mathbf{M}}$.

Structural approximation generalises algebro - geometric *deformation* and metric approximation, e.g. *Gromov-Hausdorff limit of metric spaces*

Second question: what property of the "real" **M** allows laws of physics? why do we hope that a few laws of physics can describe **M**?

Philosophers call this property **"algorithmic compressibility"**.

In model theory we have a corresponding notion **categoricity in uncountable powers**: very large structure **M** describable uniquely by its concise (countable) first order theory (and the cardinal number measuring its size).

Uncountable categoricity of **M** has strong structural consequences (stability theory). In combination with the topological assumption on **M** we come to the definition of a *Zariski Geometry*:

# Zariski geometries

Let **M** be a structure given with a family of basic relations (subsets of $M^n$) called **closed**.

We postulate for a Zariski geometry **M**:

Closed subsets form a **Noetherian Topology**

**Dimension** is assigned to any closed $S \subseteq M^n$

**Completeness:** Projections of closed are closed

**Addition formula:**

$$\dim S = \dim \mathrm{pr}(S) + \min_{a \in \mathrm{pr}(S)} \dim(\mathrm{pr}^{-1}(a) \cap S)$$

for any closed irreducible $S$.

**Pre-smoothness:** For any closed irreducible $S_1, S_2 \subseteq M^n$,

$$\dim S_1 \cap S_2 \geq \dim S_1 + \dim S_2 - \dim M^n$$

in each component.

## Known Zariski geometries

1. Smooth complete algebraic varieties over an algebraically closed field, in the natural language (1990).

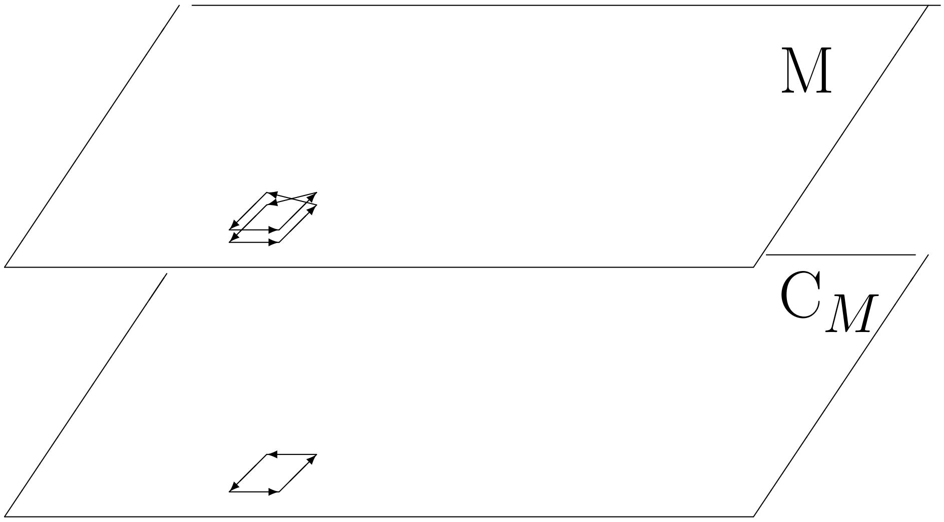
2. Compact complex manifolds, in the natural language (1993).

3. Solution spaces of well-defined systems of (partial) differential equations. (2001)

4. "New" (non-commutative) geometries (1991-...).

**Classification Theorem (Hrushovski, Z. 1993**
**and its later (Z. 2003-...) extensions.**

A typical Zariski geometry (of dimension 1) is a "noncommutative" finite cover of an algebraic variety (of dimension 1).


M
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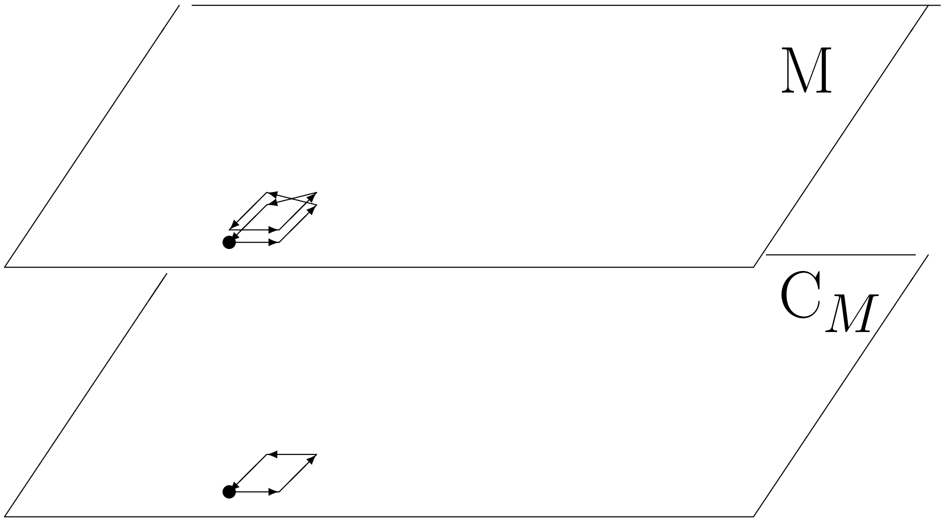
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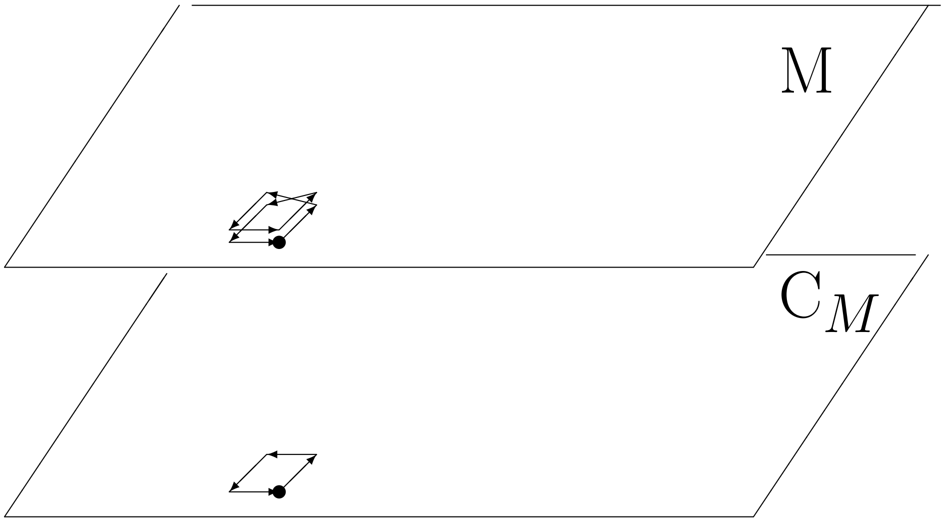
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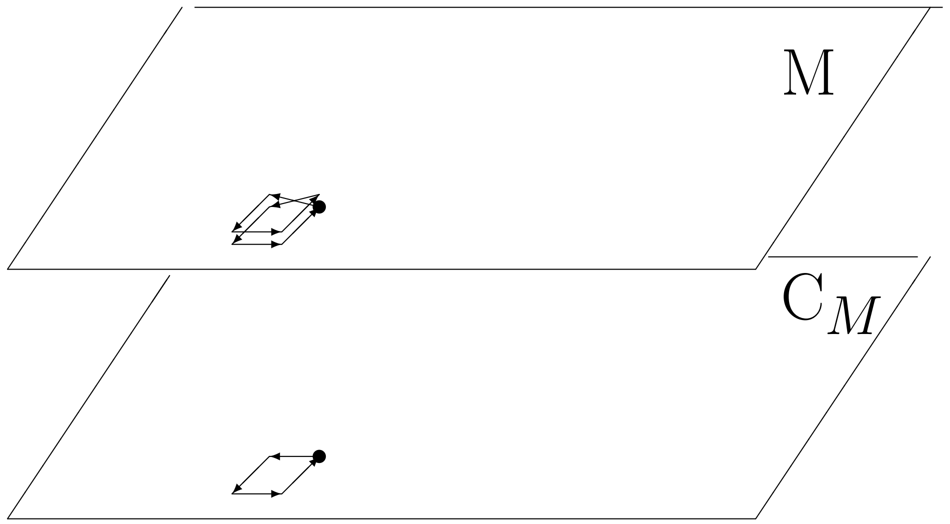
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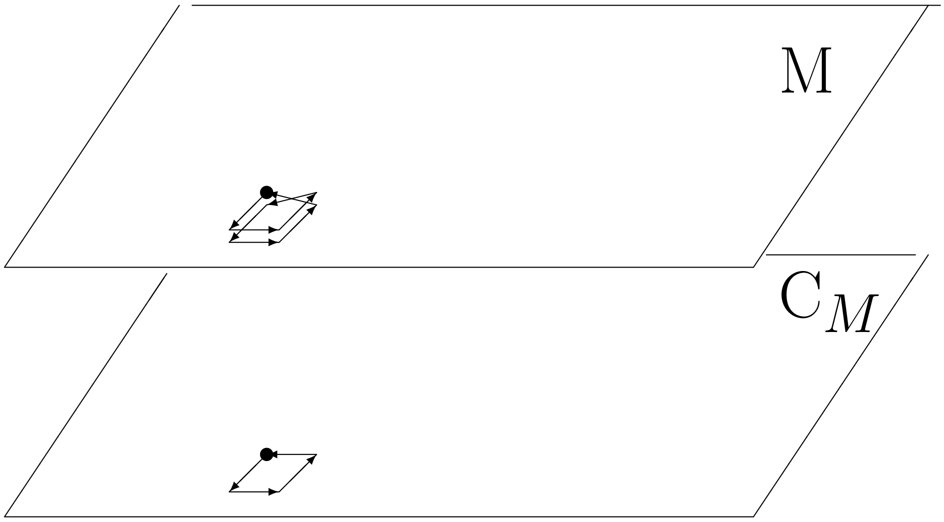
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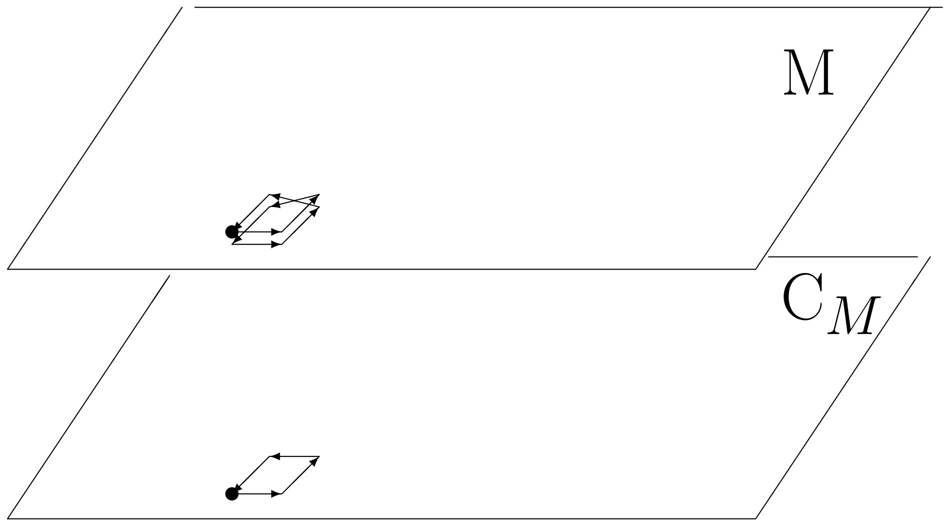
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How to recover the hidden relations?

**Use non-commutative "coordinate algebra"** (Heisenberg 1927, ...Drinfeld, Connes 1980-...)

**Theorem**(2006) Any quantum algebra $\mathcal{A}_q$ over an algebraically closed field $\mathbb{F}$ **at root of unity** $q$ gives rise to a Zariski geometry $\mathbf{M}_q$, so that $\mathcal{A}_q = \mathcal{A}(\mathbf{M}_q)$. Typically, $\mathbf{M}_q$ is non-classical, that is not an algebro-geometric object over $\mathbb{F}$.

In more general cases $\mathbf{M}_q$ is a (non-Noetherian) *analytic* Zariski geometry.

$$\mathbf{M}_q \leftrightarrow \mathcal{A}_q.$$

**Example.** The Heisenberg algebra $(\mathrm{P}, \mathrm{Q})$ with defining relation

$$\mathrm{QP} - \mathrm{PQ} = ih$$

together with the **time evolution operator**

$$e^{i\frac{H}{h}t}$$

has a full information about the quantum system with the **Hamiltonian**

$$H = \frac{\mathrm{P}^2}{2} + W(\mathrm{Q})$$

$$W(\mathrm{Q}) = 0 \quad \text{free particle}$$
$$W(\mathrm{Q}) = a\mathrm{Q}^2 \quad \text{harmonic oscillator}$$

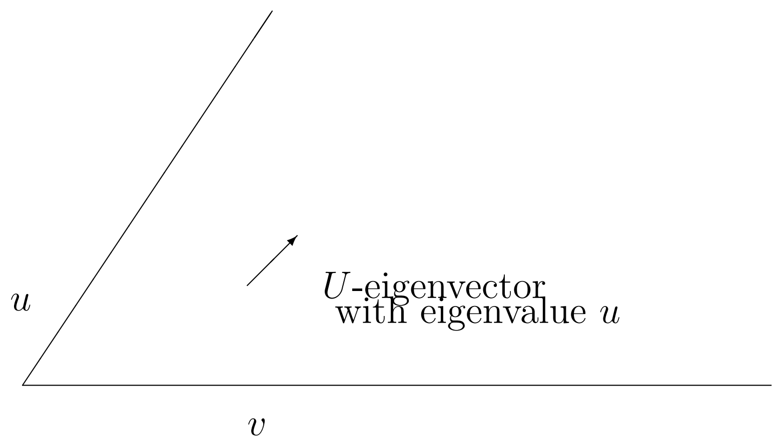
**Example continued.** Let $U = e^{i\mathrm{Q}}$, $V = e^{i\mathrm{P}}$, $q = e^{ih}$. Then

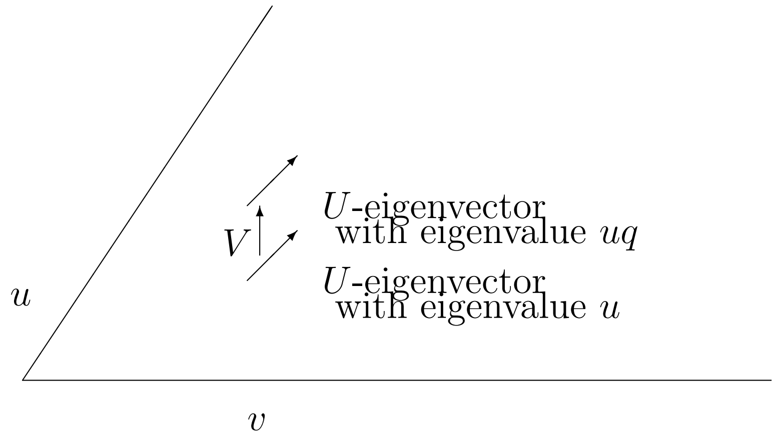
$$UV = qVU.$$

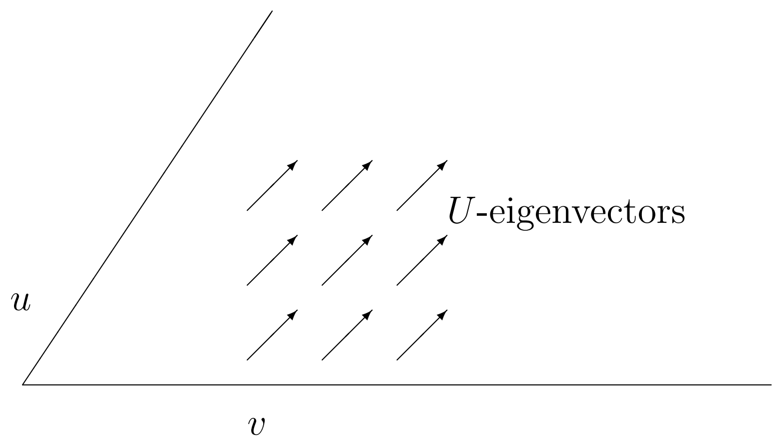
$\mathcal{A}_q$ generated by $U$ and $V$ well approximates the Heisenberg algebra.
The $\mathbf{M}_q = T_q^2(\mathbb{C})$ corresponding to $\mathcal{A}_q$ is a 2-dimensional non-classical Zariski geometry (quantum torus).
If

$$q = \epsilon = e^{i\frac{2\pi}{N}}$$

then $T_\epsilon^2(\mathbb{C})$ is a Noetherian Zariski geometry.


$u$
$U$-eigenvector
with eigenvalue $u$
$v$


U-eigenvector
with eigenvalue uq
V
U-eigenvector
with eigenvalue u
u
v


$U$-eigenvectors
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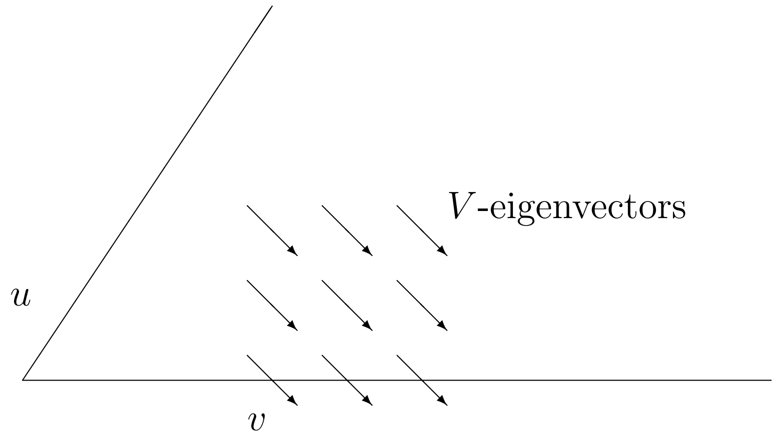
V-eigenvectors
u
v

**Theorem.** For a sequence $T^2_\epsilon(\mathbb{C})$ to approximate $T^2_q(\mathbb{C})$ it is sufficient and necessary that,
(a)

$$\lim_N \frac{2\pi}{N} = h$$

and
(b) given $m \in \mathbb{N}$, for almost all $N$ (with respect to the ultrafilter) $m|N$.

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**Corollary.** We may replace $h$ by $\frac{2\pi}{N}$ such that $m|N$ for all $m << N$.

Now we work in an irreducible $(U, V)$-module generated by $|u, v\rangle$.

$$\{|uq^k, v\rangle \quad : k = 0, 1 \ldots N - 1\}$$

forms an *orthonormal* basis of $U$-eigenvectors.

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$$|vq^m, u\rangle = \frac{1}{\sqrt{N}} \sum_{0 \le k < N} q^{-mk} |uq^k, v\rangle$$

$$|uq^k, v\rangle = \frac{1}{\sqrt{N}} \sum_{0 \le m < N} q^{km} |vq^m, u\rangle$$

$$N = \dim = \frac{2\pi}{h}.$$

**What changes if we replace $U, V$ by $U^a, V^b$, $a, b \in \mathbb{Q}$?**

Then

$$U^a V^b = q^{ab} V^b U^a$$

and the dimension $N$ of the irreducible module change

**From the condition on structural approximation the new**

$$\dim = \frac{N}{ab}.$$

Correspondingly

$$|vq^{abm}, u\rangle = \sqrt{\frac{ab}{N}} \sum_k q^{-abmk} |uq^{abk}, v\rangle$$

$$|uq^{abk}, v\rangle = \sqrt{\frac{ab}{N}} \sum_m q^{abkm} |vq^{abm}, u\rangle$$

## Time evolution for the free particle

Choose $t = \frac{m}{n}$ and add one more operator (time evolution)

$$K^t = e^{i\frac{\mathrm{P}^2}{2h}t}.$$

Then

$$K^t V K^{-t} = V \text{ and } K^t U K^{-t} = q^{\frac{t}{2}} V^t U.$$

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**Lemma** $K^t$ maps the (orthonormal) system $|u, v\rangle$ of $U$-eigenvectors to an orthonormal system of $q^{\frac{t}{2}} V^t U$-eigenvectors.

$$K^t|u,1\rangle = c_0\sqrt{\frac{t}{N}} \sum_{0\leq k<\frac{N}{t}} q^{t\frac{k^2}{2}}|uq^{-tk},1\rangle$$

$$|c_0| = 1.$$

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**Corollary.** In terms of Dirac's calculus

$$\langle x_1|K^t|x_2\rangle = c_0\sqrt{\frac{ht}{2\pi}}e^{i\frac{(x_1-x_2)^2}{2ht}}.$$

This is the *Feynman propagator for the free particle.*

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Alternatively we can express $K^t|u,1\rangle$ in terms of $|vq^{tm},u\rangle$ and then use

$$|vq^{tm},u\rangle = \sqrt{\frac{t}{N}} \sum_{0\le k<N} q^{-mk}|uq^{tk},v\rangle.$$

Thus we get another expression

$$K^t|u,1\rangle = \frac{t}{N}\sum_k\sum_p q^{-t(\frac{p^2}{2}-pk)}|uq^{kt},1\rangle.$$

Comparing the two expressions for $\langle x_1|K^t|x_2\rangle$, we get Gauss' sum

$$\sum_p q^{-t(\frac{p^2}{2}-pk)} = c_0\sqrt{\frac{N}{t}}q^{t\frac{k^2}{2}} = c_0\sqrt{\frac{2\pi}{ht}}e^{i\frac{(x_1-x_2)^2}{2ht}}$$

known to hold for even integers $\frac{N}{t}$.

$$c_0 = \frac{1+i}{\sqrt{2}}$$

This corresponds to the physics (non-convergent) integral calculation

$$\int_R e^{-a\frac{x^2}{2}-ibx}dx = \sqrt{\frac{2\pi}{a}}e^{-\frac{b^2}{2a}}, \quad a = iht$$

# Time evolution for Harmonic oscillator

$$K^t = e^{it\frac{H}{h}}.$$

Similar strategy can be applied.
First we calculate in the usual way that

$$K_H^t U K_H^{-t} = q^{\frac{\sin t \cos t}{2}} V^{\sin t} U^{\cos t}$$

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Pick up $t$ such that

$$e = \sin t, \quad f = \cos t \quad g = ef^{-1}, \qquad e, f, g \in \mathbb{Q}$$

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Direct calculation as above yield

$$\langle x_1|K^t|x_2\rangle =$$

$$= c_0\sqrt{\frac{h\cdot|\sin t|}{2\pi}}\exp i\frac{(x_1^2+x_2^2)\cos t - 2x_1x_2}{2h\sin t}$$

$$|c_0| = 1.$$

Or, in a different normalisation (if we replace sums by integrals)

$$c_0\frac{1}{\sqrt{2\pi h\cdot|\sin t|}}\exp i\frac{(x_1^2+x_2^2)\cos t - 2x_1x_2}{2h\sin t}$$

*Feynman propagator for the Harmonic oscillator.*

$$\frac{1}{\sqrt{2\pi h\cdot|\sin t|}}$$

*quantum fluctuations*, equal, by our calculation, to the dimension of the correspondent irreducible module.

# References

1. B.Zilber, The noncommutative torus and Dirac calculus (draft).
http://people.maths.ox.ac.uk/ zilber/dirac.dvi

2. B.Zilber, Notes on Approximation (draft)
http://people.maths.ox.ac.uk/ zilber/approx.dvi

3. B.Zilber, *A class of quantum Zariski geometries.* In: Model Theory with applications to algebra and analysis, I and II (Z. Chatzidakis, H.D. Macpherson, A. Pillay, A.J. Wilkie editors), Cambridge University Press, Cambridge.

4. B.Zilber, *Non-commutative Zariski geometries and their classical limit.* arXiv0707.0780[math.QA] June 2007