Axiomatic Set Theory: Problem sheet 4

4 **1.** Prove that $\forall\alpha, \beta \in On$, (i) $V_\alpha \cap On = \alpha$, and (ii) if $\alpha \in V_\beta$, then $V_\alpha \in V_\beta$.

2. Complete the proof of Lévy's Reflection Principle.

3. A *club* is, by definition, a closed, unbounded class of ordinals. Prove that if U_1 and U_2 are clubs then so is $U_1 \cap U_2$. More generally, suppose that X is a class such that $X \subseteq \omega \times On$. For $i \in \omega$, let $X_i = \{\alpha \in On : \langle i, \alpha\rangle \in X\}$. Suppose that for all $i \in \omega$, X_i is a club. Prove that $\bigcap_{i\in\omega} X_i$ is a club.

4. (i) It is known that there is a formula $\phi(x)$ of LST (without parameters) such that (in ZF one can prove that) for any set a, $\phi(a)$ iff "$\langle a, \in\rangle \models$ ZF and a is transitive". Further, this formula is A-absolute for any transitive class A (see sheet 3, question 4). Show that one cannot prove the sentence $\exists x\phi(x)$ from ZF. [Hint: Consider the least $\alpha \in On$ such that $\exists x \in V_\alpha(\phi(x))$.]

(ii) As formulated in the lectures, ZF is a countably infinite collection of axioms (since there is one separation and replacement axiom for each formula of LST, and there are clearly a countably infinite number of such formulas). Prove that there is no finite subcollection, T, say, of ZF, such that $T \vdash$ ZF.

5. * What is wrong with the following argument:

Let $\{\sigma_i : i \in \omega\}$ be an enumeration of all the axioms of ZF. By Lévy's Reflection Principle, for each $i \in \omega$, the class $\{\alpha \in On : \langle V_\alpha, \in\rangle \models \sigma_i\}$ (call it X_i) is a club (since $(V, \in) \models \sigma_i$). By question (3) above, $\bigcap_{i\in\omega} X_i$ is a club (we are using question (3) by setting $X = \{\langle i, \alpha\rangle : \alpha \in X_i\}$). In particular, $\bigcap_{i\in\omega} X_i$ is non-empty. Let $\beta \in \bigcap_{i\in\omega} X_i$. Then $\beta \in X_i$ for all $i \in \omega$, so $\langle V_\beta, \in\rangle \models \sigma_i$ for all $i \in \omega$, so $\langle V_\beta, \in\rangle \models$ ZF. Hence $\phi(V_\beta)$ holds, so $\exists x\phi(x)$ (where $\phi(x)$ is the formula in (4)(i)). Since $(V, \in)$ is an arbitrary model of ZF, we have ZF $\vdash \exists x\phi(x)$!