Axiomatic Set Theory: Problem sheet 7

1. Prove that for any infinite cardinal κ, $cf(\kappa)$ is a regular cardinal.

2. Suppose κ, λ are infinite cardinals such that $\kappa \geq \lambda$. Prove that if $\lambda \geq cf(\kappa)$, then $\kappa^\lambda > \kappa$. Suppose now that $\lambda < cf(\kappa)$, and that κ has the property that for any cardinal μ, if $\mu < \kappa$ then $2^\mu \leq \kappa$. Prove that $\kappa^\lambda = \kappa$. Hence show that if GCH is assumed, then for any infinite cardinals κ, λ with $\kappa \geq \lambda$, we have $\kappa^\lambda = \kappa$ or κ^+.

3. Suppose κ is an *uncountable regular* cardinal. Let $g : \kappa \to \kappa$ be any function. Prove that for any $\alpha < \kappa$, there exists $\beta < \kappa$, with $\alpha \leq \beta$, such that β is closed under g (ie. for all $\gamma < \beta$, $g(\gamma) < \beta$).

4. (Optional) Let κ be an uncountable regular cardinal with the property that for any cardinal $\mu < \kappa$, we have $2^\mu < \kappa \ldots (*)$.
Prove that (i) if α is any cardinal and $\alpha < \kappa$, then $|V_\alpha| < \kappa$, (ii) $|V_\kappa| = \kappa$, (iii) if κ is regular, then $\langle V_\kappa, \in \rangle \vDash$ ZFC.
(For (iii) you need consider only the replacement scheme, since we essentially showed that if α is a limit ordinal and $\alpha > \omega$, then $\langle V_\alpha, \in \rangle$ satisfies all the axioms of ZFC except, possibly, replacement.)
Deduce that in ZFC one cannot prove the existence of a cardinal that satisfies $(*)$.