# On model theory, non-commutative geometry and physics

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## 1 Introduction

**1.1** Our motivation for working on the subject presented below comes from the realisation of the rather paradoxical situation with the mathematics used by physicists in the last 70 or so years. Physicists have always been ahead of mathematicians in introducing and testing new methods of calculations, leaving to mathematicians the task of putting the new methods and ideas on a solid and rigorous foundation. But this time, with developments in quantum field theory huge progress achieved by physicists in dealing with singularities and non-convergent sums and integrals (famous Feynman path integrals) has not been matched so far, after all these years, with an adequate mathematical theory. A nice account of some of these methods with a demonstration of challenging calculations can be found in [6], and more detailed account in [7] (Chapter 2. *The basic strategy of exctracting finite information from infinities*).

One may suggest that the success in developing this calculus in the absence of a rigorous mathematical theory is due to the fact that the physicist, in fact, uses an implicit or explicit knowledge of the structure of his model which is not yet available in mathematical terms. A beautiful and honest account by a mathematician attemting to translate physicists' vision into a mathematical concept provides the introductory section of [10]. In particular, many formulas which to the mathematician's eye are defined in terms of a metric or a measure are not what they look. Typically, it is crucial that discrete approximations to the continuous models have the same type of symmetries or other structural properties, though the formula does not say this.

**1.2** Model theory and logic in general has an obvious advantage over general mathematics in this situation. The logician is not restricted by any conventional mathematics and is ready to deal with any type of structure at all. Moreover, modern model theory has worked out, in fact, a very efficient hierarchy of types of structures (stability theory and beyond), and has a crucial experience in introducing new tailor-made structures to accommodate and deal with specific mathematical problems. One particularly relevant class of structures was discovered by Hrushovski and the author [5] in an attempt to identify and characterise, essentially, "logically perfect" structures, or more technically, the top level of the stability hierarchy. These are *Zariski structures* (in some variation called also *Zariski geometries*), defined in very general terms of geometric flavour (see below) and modelled on algebraic varieties over algebraically closed fields equipped with relations corresponding to Zariski closed sets. The one-dimensional objects in this class have been characterised as finite covers of algebraic curves, which is generally quite satisfactory and has led to important applications. But the structure "hidden" in the finite covers makes, in general, the object not definable in the underlying algebraically closed field. The analysis of these in [2], [4] and [3] shows that these "nonclassical" structures are of noncommutative geometric origin, essentially of the same nature as structures of quantum physics.

This link between model theory of Zariski structures, noncommutative geometry and physics is essential part of what is presented in the article.

**1.3** The process of understanding the physical reality by working in an **ideal** model can be interpreted as follows. We assume that the ideal model $\mathbf{M}_{\rm ideal}$ is being chosen from a class of "nice" structures, which allows a good theory. We suppose that the real structure $\mathbf{M}_{\rm real}$ is "very similar" to $\mathbf{M}_{\rm ideal}$, meaning by this that the description of $\mathbf{M}_{\rm ideal}$ (as a set of statements) has a large intersection with the corresponding description of $\mathbf{M}_{\rm real}$. The notion of "large" is of course relative and can be formalised dynamically, by assuming that $\mathbf{M}_{\rm ideal}$ is approximated by a sequence $\mathbf{M}_i$ of structures and $\mathbf{M}_{\rm real}$ is one of these, $\mathbf{M}_i = \mathbf{M}_{\rm real}$, sufficiently close to $\mathbf{M}_{\rm ideal}$. The notion of approximation must also contain both logical (qualitative) and topological ingredients. Topology gives us a way to speak of "nearness" between points and events. Naturally, the reason that we wouldn't distinguish two points in the ideal model $\mathbf{M}_{\rm ideal}$ is that the corresponding points are very close in the real world $\mathbf{M}_{\rm real}$, so that we do not see the difference (using the tools available). In the limit of the $\mathbf{M}_i$'s this sort of difference will manifest itself as an infinitesimal. In other words, the limit passage from the sequence $\mathbf{M}_i$ to the ideal model $\mathbf{M}_{\rm ideal}$ must happen by killing the infinitesimal differences. This corresponds to taking a specialisation ("equations" preserving map) from an ultraproduct $\prod_D \mathbf{M}_i$ to $\mathbf{M}_{\rm ideal}$.

We formalise and study the notion of a *structural approximation* below. We give a number of examples that demonstrate that this notion covers well-known notions of approximation such as:

- limit point in a topological space;
- Gromov-Hausdorff limit of metric spaces;
- deformation of algebraic varieties

We note that the scheme is quite delicate regarding metric issues. In principle we may have a well-defined metric, agreeing with the qualitative topology, on the ideal structure only. Existence of a metric, especially the one that gives rise to a structure of a differentiable manifold, is one of the key reasons of why we regard some structures as 'nice' or 'tame'. The problem of whether and when a metric on $\mathbf{M}$ can be passed to approximating structures $\mathbf{M}_i$ might be difficult, indeed we don't know how to answer this problem in some interesting cases.

**1.4** In the next section we explain how Zariski geometries, introduced and studied by model-theorists can be used, via noncommutative geometry, to model physical spaces.

In section 3 we introduce one special type of Zariski geometries corresponding to quantum 2-tori (*geometric quantum 2-tori*) and in section 4 structural approximation in this class of geometries is studied. It is shown that a general geometric quantum 2-tori can be structurally approximated by quantum 2-tori *at roots of unity.* This is used in section 5 to rigorously calculate kernels of Feynman propagators for two cases: the free particle and the quantum harmonic oscillator.

# 2 Structures of physics and model theory

**2.1** How should a logician think of a structure for physics? We may try to stick to model-theoretic tradition and imagine the physics universe in a form of a structure $\mathbf{M}$ with a domain $M$ and a collection of basic predicates $\mathcal{C}$,

$$\mathbf{M} = (M, \mathcal{C}).$$

We have to accept that for the physicist it is very sensitive to distinguish between a formula of the form, say, $R$ and $\neg R$. In fact the physicist thinks, often implicitly, of relations of the form of equation

$$R_{f,g}(x_1, \ldots, x_n) : \ f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$$

for certain nice functions, or inequalities

$$R_{f,r}(x_1, \ldots, x_n) : \ |f(x_1, \ldots, x_n)| \leq r.$$

So our collection $\mathcal{C}$ must be generating a topology on $M^n$, for every cartesian power $M^n$ of the universe ( a basis of closed sets of the topology), with the assumption that the projections (permutations) $\mathrm{pr}_{i_1 \ldots i_k} : M^n \to M^k$, $(x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_k})$ are continuous. Such a structure $\mathbf{M} = (M, \mathcal{C})$ we call a **topological structure** in [1].

Furthermore, the physicist would want to think that the closed sets are lines, surfaces and so on, that is we should be able to assign a dimension, $\dim R$, to every closed set $R$ and its projections, with certain nice properties, e.g. for an irreducible closed $R \subseteq M^n$,

$$\dim R = \dim \mathrm{pr}\, R + \dim\{\text{generic fiber}\}.$$

Finally, and very importantly, there must be a meaningful notion of smoothness expressible in terms of the structure. For topological structures with a good dimension such a notion, called *presmoothness*, has been found in [5] (we use the terminology of [1]). An open irreducible set $U$ is said to be **presmooth** if, for any irreducible relatively closed subsets $S_1, S_2 \subset U$, and any irreducible component $S_0$ of the intersection $S_1 \cap S_2$,

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim U.$$

Taken together these assumptions lead to a definition of a **Zariski structure** (Zariski geometry), in its Noetherian version as in [5], or a more general one, called **analytic Zariski structure** as in [1].

**2.2** The paper [5] proves an important classification theorem for a one-dimensional Zariski geometry $\mathbf{M}$. In the most interesting *non-locally modular* case *there is a smooth algebraic curve* $C$ *over an algebraically closed field* $\mathbb{F}$ *and a surjective map* $p : M \to C(\mathbb{F})$, *all definable in* $\mathbf{M}$, *so that the fibres of* $p$ *are uniformly finite.*

This almost reduces Zariski geometry to algebraic geometry, but in fact the structure hidden in the finite fibres can be quite complicated and interesting on its own, and this in general can not be definable in the field $\mathbb{F}$. Roughly speaking, there are not enough definable *coordinate functions* $M \to \mathbb{F}$ to encode all the structure on $M$. The usual coordinate algebra gives us just $C(\mathbb{F})$ and the rest of the structure $\mathbf{M}$ remains hidden.

We have shown in [2] that this difficulty can be overcome if one introduces an appropriate *coordinate algebra of operators* $\mathcal{A}(\mathbf{M})$, generally noncommutative. This analysis strongly resembles the process that lead physicists of the 1920th towards the introduction of what nowadays will be identified as *noncommutative geometry.*

**2.3** In [4] we continued our investigation of possible links between noncommutative geometry and Zariski structures and developed a construction which, given an algebraically closed field $\mathbb{F}$ and a *quantum* $\mathbb{F}$*-algebra* $\mathcal{A}$ *at root of unity,* produces a Noetherian Zariski structure $\mathbf{M}$. By a reverse correspondence one can recover $\mathcal{A}$ as a coordinate algebra of $\mathbf{M}$. In most cases $\mathbf{M}$ is *non-classical*, that is not definable in $\mathbb{F}$ or indeed in any algebraically closed field. But when $\mathcal{A}$ is commutative $\mathbf{M}$ is just the affine variety corresponding to the affine commutative algebra $\mathcal{A}$.

$$\begin{array}{ccc} \mathbf{M} & \longleftrightarrow & \mathcal{A} \\ Zariski & & operator \\ geometry & & algebra \end{array}$$

Although [4] developes a systematic procedure only for $\mathcal{A}$ at root of unity, the same or very similar construction produces Zariski geometries (as one can see in [2] and [3]) from more general quantum algebras. We do not have precise conditions of when this scheme works but it does in most important cases. One of the cases is discussed below, section 3.

When $\mathcal{A}$ is an operator algebra describing essential physical process, $\mathbf{M}$ could be regarded as the corresponding geometric space where the process takes place. This was the case in Newtonian physics, the luxury that was lost with the advent of quantum physics.

**2.4** At this point we would like to mention a different but rather similar in spirit project undertaken by Isham, Butterfield and Doering in a series of papers (see e.g.[14], [15]). The aim of their project is to introduce a mathematical object "that takes the role of the state space of a quantum system... Here the state space is to be seen in analogy to classical physics and does not mean Hilbert space." The main idea in this approach is to use topoi in place of the category of sets and classical structures. Isham and Doering construct a topos associated to a given noncommutative algebra. We remark here that although technically very different, our construction of a Zariski geometry associated to a given noncommutative algebra can be seen in a similar light. We also remark that our approach does not reject completely the Hilbert space paradigm but rather the Zariski structure emerges as an object embedded in a Hilbert space. For the reader with a knowledge of the model theory of differentially closed fields the reference to the following can be helpful, in order to get a better idea on the sort of connection between a Hilbert space and a Zariski structure. Recall A.Pillay's theorem [16] that states that finite Morley rank substructures of a differentially closed field $\mathcal{F}$ can be seen as Zariski structures in a natural Zariski topology. Now note that $\mathcal{F}$ can be thought as an abstraction for a field of functions, an infinite dimensional space over the field of constants $\mathbb{C}$.

# 3 Noncommutative 2-tori

**3.1** Let $\mathbb{F}$ be an algebraically closed field. Consider an $\mathbb{F}$-algebra $\mathcal{A}_q$ generated by "operators" $U, U^{-1}, V, V^{-1}$ satisfying the relation

$$VU = qUV, \tag{1}$$

for some $q \in \mathbb{F}^*$. The algebra $\mathcal{A}_q$ is called a *quantum (or noncommutative) 2-torus.* When $q$ is a root of unity, $\mathcal{A}_q$ is one of the basic examples of a quantum algebra at root of unity and the construction of a structure associated to $\mathcal{A}_q$ presented below is a modification and generalisation of the one in [4].

We denote

$$\mathcal{G} = \{q^m : m \in \mathbb{Z}\},$$

a multiplicative subgroup of $\mathbb{F}^*$.

Fix a pair $(u, v) \in \mathbb{F}^* \times \mathbb{F}^*$. We will associate to $(u, v)$ two $\mathcal{A}_q$-modules $M_{|u,v>}$ and $M_{<v,u|}$, which in case of $q$ root of unity can be identified as two representations of the same module.

The module $M_{|u,v>}$ generated by elements labeled $\{\mathbf{u}(\mathrm{g}u,v) \ : \ \mathrm{g}\in\mathcal{G}\}$ satisfying

$$\begin{array}{l} U:\mathbf{u}(\mathrm{g}u,v)\mapsto \mathrm{g}u\,\mathbf{u}(\mathrm{g}u,v)\\ V:\mathbf{u}(\mathrm{g}u,v)\mapsto v\,\mathbf{u}(q^{-1}\mathrm{g}u,v)\end{array} \tag{2}$$

Define also the module $M_{<v,u|}$ generated by $\{\mathbf{v}(\mathrm{g}v,u) \ : \ \mathrm{g}\in\mathcal{G}\}$ satisfying

$$\begin{array}{l} U:\mathbf{v}(\mathrm{g}v,u)\mapsto u\,\mathbf{v}(q\mathrm{g}v,u)\\ V:\mathbf{v}(\mathrm{g}v,u)\mapsto \mathrm{g}v\,\mathbf{v}(\mathrm{g}v,u)\end{array} \tag{3}$$

Eventually we would like to see the both modules as submodules of an ambient module, which in case $q$ is a root of unity will coincide with each of the modules, that is $M_{|u,v>}=M_{<v,u|}$ in this case.

Now let $\phi:\mathbb{F}^*/\mathcal{G}\to\mathbb{F}^*$ be a (non-definable) "choice function" which chooses an element in every class $u\mathcal{G}$ and $\Phi=\mathrm{range}\,\phi$, a set of representatives of $\mathbb{F}^*/\mathcal{G}$. We will work with $\Phi^2=\Phi\times\Phi$, a set of representatives of $\mathbb{F}^*/\mathcal{G}\times\mathbb{F}^*/\mathcal{G}$.

Let

$$\mathbb{U}_\phi:=\{\mathrm{g}_1\cdot\mathbf{u}(\mathrm{g}_2u,v):\langle u,v\rangle\in\Phi^2,\mathrm{g}_1,\mathrm{g}_2\in\mathcal{G}\}$$

a subset of

$$\bigcup_{u,v}M_{|u,v>}$$

and

$$\mathbb{V}_\phi:=\{\mathrm{g}_1\cdot\mathbf{v}(\mathrm{g}_2v,u):\langle u,v\rangle\in\Phi^2,\mathrm{g}_1,\mathrm{g}_2\in\mathcal{G}\},$$

a subset of

$$\bigcup_{v,u}M_{<v,u|}$$

Our language for the structure will have unary predicates $\mathbb{U}$ and $\mathbb{V}$ for the sorts $\mathbb{U}_\phi$ and $\mathbb{V}_\phi$.

We will also consider the sets

$$\mathbb{F}\mathbb{U}_\phi:=\{x\cdot\mathbf{u}(\mathrm{g}u,v):\langle u,v\rangle\in\Phi^2,x\in\mathbb{F},\mathrm{g}\in\mathcal{G}\}$$
$$\mathbb{F}\mathbb{V}_\phi:=\{y\cdot\mathbf{v}(\mathrm{g}v,u):\langle u,v\rangle\in\Phi^2,y\in\mathbb{F},\mathrm{g}\in\mathcal{G}\},$$

which can be defined as sets of pairs factored by an equivalence relation that identifies $\mathrm{g}\in\mathcal{G}$ as an element of $\mathbb{F}^*$ :

$$\mathbb{F}\mathbb{U}_\phi=(\mathbb{F}\times\mathbb{U}_\phi)/E,\quad \mathbb{F}\mathbb{V}_\phi=(\mathbb{F}\times\mathbb{V}_\phi)/E.$$

We consider the **pairing**

$$\langle \cdot \mid \cdot \rangle : \ \mathbb{U}_\phi \times \mathbb{V}_\phi \to \mathbb{F}$$

(as a ternary relation on $\mathbb{U}\times\mathbb{V}\times\mathbb{F}$) defined as follows. For $q^s\,\mathbf{v}(q^m v, u) \in \mathbb{U}_\phi$ and $q^r\,\mathbf{u}(q^k u, v) \in \mathbb{V}_\phi$ set

$$\langle q^s\,\mathbf{v}(q^m v, u) | q^r\,\mathbf{u}(q^k u, v)\rangle = q^{-km-s+r}, \tag{4}$$

and $\langle q^s\,\mathbf{v}(v', u) | q^r\,\mathbf{u}(u', v)\rangle$ not defined (the ternary relation does not hold) if $v' \notin \mathcal{G}\cdot v \ \vee \ u' \notin \mathcal{G}\cdot u$.

Our structure now consists of 3 sorts $\mathbb{U}_\phi$, $\mathbb{V}_\phi$ and $\mathbb{F}$ with the structure of field on $\mathbb{F}$, the operators $U$ and $V$ acting on the sorts as described above and the pairing. We will call the structure $(\mathbb{U}_\phi, \mathbb{V}_\phi, \mathbb{F})$ a **(geometric) quantum 2-torus** associated with the $\mathcal{A}_q$ above.

**3.2 Theorem** *Given an algebraically closed field* $\mathbb{F}$ *with* $q, H \in \mathbb{F}$, *choice functions* $\phi$ *and* $\psi$, $\mathbb{F}^*/\mathcal{G} \to \mathbb{F}^*$ *and two corresponding quantum tori* $(\mathbb{U}_\phi, \mathbb{V}_\phi, \mathbb{F})$, *and* $(\mathbb{U}_\psi, \mathbb{V}_\psi, \mathbb{F})$, *there exists an isomorphism over* $\mathbb{F}$ *between the two structures.*

*In other words, the definition does not depend on the choice function and the isomorphism type of the structure is determined by the isomorphism type of the field* $\mathbb{F}$ *and constants* $q$ *and* $H$.

*For* $q$ *root of unity the geometric quantum torus is a Noetherian Zariski structure. (In the general case we expect it to be an analytic Zariski structure.)*

*If* $q \neq 1$, *the structure is not definable in* $(\mathbb{F}, \mathcal{G})$ *(the field with a predicate for the subgoup).*

The case of $q$ root of unity is proved in [4]. The general case is similar.

Following the theorem we omit mentioning the choice function in the construction of the geometric tori and refer to these as simply $(\mathbb{U}, \mathbb{V}, \mathbb{F})$.

**3.3** The quantum torus operators were introduced to physics by H.Weyl, with $\mathbb{F} = \mathbb{C}$, the complex numbers,

$$q = e^{2\pi i\hbar},$$

$\hbar$ the (reduced) Planck constant, $V = \exp iP$, $\ U = \exp 2\pi i Q$, for $P, Q$ the momentum and position operators satisfying the Heisenberg commutation relation

$$QP - PQ = i\hbar I \tag{5}$$

($I$ the identity) from which the Weyl commutation relation (1) follows.

Clearly $q$ is a root of unity of order $N$ if

$$q = e^{\frac{2\pi i n}{N}}, \;\; \hbar = \frac{n}{N}, \;\; n \in \mathbb{N}, \;\; (n, N) = 1.$$

We assume from now on that $\mathbb{F} = \mathbb{C}$ and our parameters are chosen as in this subsection.

**3.4** Note that in the case $q$ is a root of unity of order $N$ the pairing (4) gives rise to an inner product $(\cdot \mid \cdot)$ defined by skew-linearity from

$$\big(\mathbf{u}(q^k u, v) \,|\, \mathbf{v}(q^m v, u)\big) := \frac{1}{\sqrt{N}} \langle \mathbf{v}(q^m v, u) | \mathbf{u}(q^k u, v) \rangle = \frac{1}{\sqrt{N}} q^{-km}, \tag{6}$$

$$\big(\mathbf{v}(q^k v, u) \,|\, \mathbf{v}(q^m v, u)\big) = \delta_{km}, \tag{7}$$

$$\big(\mathbf{u}(q^k u, v) \,|\, \mathbf{u}(q^m v, u)\big) = \delta_{km}. \tag{8}$$

Observe that (6)–(8) can be read as saying that in the module $M_{|u,v\rangle} = M_{<v,u|}$

$$\mathbf{u}(q^k u, v) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} q^{-km} \mathbf{v}(q^m v, u), \tag{9}$$

$$\mathbf{v}(q^m v, u) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} q^{km} \mathbf{u}(q^k u, v), \tag{10}$$

transition formulas between two orthonormal bases.

**3.5** In the general case, when $q$ is not a root of unity, physicists work with the pairing and assume in analogy to (7), (8)

$$\begin{aligned} \langle \mathbf{v}(q^k v, u) \,|\, \mathbf{v}(q^m v, u) \rangle &= \delta(k - m), \\ \langle \mathbf{u}(q^k u, v) \,|\, \mathbf{u}(q^m v, u) \rangle &= \delta(k - m), \end{aligned}$$

where the Kronecker delta function is replaced by the Dirac delta function.

For the root of unity case these correspond to

$$\begin{aligned} \langle \mathbf{v}(q^k v, u) \,|\, \mathbf{v}(q^m v, u) \rangle &= N\delta_{km}, \\ \langle \mathbf{u}(q^k u, v) \,|\, \mathbf{u}(q^m v, u) \rangle &= N\delta_{km}. \end{aligned} \tag{11}$$

**3.6 Remark** Finally note that our structure should be considered a **complexification** of the quantum 2-torus, since we have not fully taken into account the extra condition, relevant to physics, that $U$ and $V$ are unitary.

**3.7 The $*$-conditions and the case $q = 1$.** In noncommutaive geometry an important role is played by the extra $*$-conditions. In case of the noncommutative tori it is the assumption that $U$ and $V$ must be considered as unitary operators. This will have the consequence that the eigenvalues of $U$ and $V$ are on the unit circle $\mathcal{S}$ (complex numbers of modulus 1) and so we will be forced to consider just the substructure of $(\mathbb{U}_\phi, \mathbb{V}_\phi, \mathbb{C})$ with $\Phi \subseteq \mathcal{S}$.

In case $q = 1$ (without $*$-assumptions) $\Phi = \mathbb{C}^*$ and every element of $\mathbb{U}_\phi$ as well as $\mathbb{V}_\phi$ is determined just by $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$. So, in fact, each of these sorts can be simply identified with the algebraic 2-torus $\mathbb{C}^* \times \mathbb{C}^*$ (we can say that the bundles are trivial) and the actions of $U$ and $V$ are definable in the usual algebraic (Zariski) structure on $\mathbb{C}^*$.

Under the assumption that $U$ and $V$ are unitary we will get the corresponding classical structure on $\mathcal{S}^2$.

# 4 Structural approximation

**4.1 General scheme.** We work in the context of topological structures as in 2.1. We say that a topological structure $\mathbf{M} = (M, \mathcal{C})$ is **complete** if, for each $S(x, y) \in \mathcal{C}$ there is $P(y) \in \mathcal{C}$ such that $\mathbf{M} \vDash \exists x S \equiv P$. We say $\mathbf{M}$ is **quasi-compact** (often just **compact**) if $\mathbf{M}$ is complete, every point in $M$ is closed and for any filter of closed subsets of $M^n$ the intersection is nonempty.

*Given a topological structure* $\mathbf{M}$ *and a family of structures* $\mathbf{M}_i$, $i \in I$, *in the same language, we say that* $\mathbf{M}$ *is* **approximated** *by* $\mathbf{M}_i$ *along an ultrafilter* $D$ *on* $I$, *if for some elementary extension* $\mathbf{M}^D \succcurlyeq \prod \mathbf{M}_i / D$ *of the ultraproduct there is a surjective homomorphism*

$$\lim_D : \mathbf{M}^D \to \mathbf{M}.$$

**Remark 1.** Assuming the continuum hypothesis in all cases of interest to us it would suffice to consider for a nonprincipal ultrafilter $D$ just $\mathbf{M}^D = \prod \mathbf{M}_i / D$.

**Remark 2.** We allow 0-ary predicates in $\mathcal{C}$, which can be interpreted as unary predicates on $I$, defining closed subset of the latter.

It is not difficult to see the following

**4.2 Proposition.** *Suppose every point of* $\mathbf{M}$ *is closed and* $\mathbf{M}$ *is approximated by the sequence* $\{\mathbf{M}_i = \mathbf{M} : i \in I\}$ *for some* $I$ *along an ultrafilter* $D$ *on* $I$, *with* $\mathbf{M}^D$ $\aleph_0$*-saturated. Then* $\mathbf{M}$ *is quasi-compact.*

According to the proposition we will normally consider only approximations to quasi-compact structures.

Nevertheless, in many cases we have to deal with an $\mathbf{M}$ which is not quasi-compact. Then we aim to construct a **compactification** $\bar{\mathbf{M}}$, a superstructure of $\mathbf{M}$, which is quasi-compact and definable in $\mathbf{M}$. So we reduce the problem of approximation to $\mathbf{M}$ to that of $\bar{\mathbf{M}}$.

**4.3 Examples.** The following two examples are discussed in [12]

1. Let $M$ be a metric space. Consider, for every positive rational number $r$ the binary relations $d_r^{\leq}(x,y)$ and $d_r^{\geq}(x,y)$, with the interpretation $\mathrm{dist}(x,y) \leq r$ and $\mathrm{dist}(x,y) \geq r$ correspondingly. The sets given by positive existential formulas in this language form our class $\mathcal{C}$.

**4.4 Proposition.** Assume $M$ and $M_i$, $i \in I$, are compact metric spaces and $M$ is the Gromov-Hausdorff limit of metric spaces along a non-principal ultrafilter $D$ on $I$.

Then, for structures $\mathbf{M} = (M, \mathcal{C})$ and $\mathbf{M}_i = (M_i, \mathcal{C})$,

$$\mathbf{M} = \lim_D \mathbf{M}_i,$$

2. Let $\mathbb{Z} = (\mathbb{Z}, +)$ be the additive group of integers and $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$, the structure with the ternary relation "$x + y = z$" interpreted as usual on $\mathbb{Z}$ and

- "$x - \infty = -\infty$" for all $x \in \mathbb{Z}$;
- "$x + \infty = +\infty$" for all $x \in \mathbb{Z}$;
- "$+\infty - \infty = x$" for all $x \in \bar{\mathbb{Z}}$.

We set $\mathcal{C}$ to consist of the relation "$x + y = z$", the unary relations $x = n$ for every integer $n$ and $+\infty$, $-\infty$, and all positive existential formulas in this language. It is easy to see that $(\bar{\mathbb{Z}}, \mathcal{C})$ is quasi-compact. This is one of the possible compactifications of $\mathbb{Z}$.

The same language makes sense for the residue classes modulo $n$ extended by $+\infty$ and $-\infty$ : $\bar{\mathbb{Z}}/n\bar{\mathbb{Z}}$.

Below we say that an ultrafilter $D$ on positive integers $\mathbb{N}$ is **profinite** if for every positive integer $m$ the subset $m\mathbb{N}$ belongs to $D$.

**4.5 Proposition.** Given an ultrafilter $D$ on positive integers $\mathbb{N}$, $\bar{\mathbb{Z}} = \lim_D \bar{\mathbb{Z}}/n\bar{\mathbb{Z}}$ if and only if $D$ is profinite.

Now we are ready to consider the structural approximation of a quantum torus at $q$ by quantum tori at roots of unity.

First we determine a compactification $(\bar{\mathbb{U}}, \bar{\mathbb{V}}, \bar{\mathbb{C}})$ of a quantum torus $(\mathbb{U}, \mathbb{V}, \mathbb{C})$ described in section 3. The compactification of $\mathbb{C}$ is standard, $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the projective complex line. The compactification of $\mathbb{U}$ and $\mathbb{V}$ is determined once we determine the compactification of the group $\mathcal{G} = q^{\mathbb{Z}}$. The latter is isomorphic to the additive group $\mathbb{Z}$, if $q$ is not a root of unity, and to $\mathbb{Z}/n\mathbb{Z}$ if $q$ is a root of unity of order $n$. We apply the compactification of the example above.

This allows us to apply the notion of structural approximation. Note that physics also requires that $q_N = \exp \frac{2\pi i M}{N}$ metrically converge to $q = \exp 2\pi i \hbar$ ($M$ and $N$ positive coprime integers) if we approximate the corresponding quantum structures. We incorporate this condition into our notion of structural approximation in the family of quantum tori by including into the language $\mathcal{C}$ for these structures the 0-ary predicates $d_r^{\leq}(1, q)$ and $d_r^{\geq}(1, q)$ which assess the distance between 1 and the constant $q$. Equivalently, for $\frac{M}{N} < 1$ and $\hbar < 1$, these assess the values of the real numbers $\frac{M}{N}$ and $\hbar$.

**4.6 Theorem.** *In order for a geometric quantum torus at generic $q$ to be approximated by quantum tori at roots of unity* $\exp \frac{2\pi i M_N}{N}$, $N \in \mathbb{N}$, *along an ultrafilter $D$ on $\mathbb{N}$ it is necessary and sufficient that $D$ is profinite and $\frac{M_N}{N}$ metrically converge to $\hbar$ along $D$.*

*In case $q = 1$ the necessary and sufficient condition is that $\frac{M_N}{N}$ converge to* 1. *In particular, in this case the geometric tori at roots of unity approximate the classical torus $\mathcal{S}^2$.*

The proof of this theorem in [12] is built on the extension of the argument in the proof of 4.5.

**4.7** A practical corollary of the theorem is that in the context of quantum tori $\hbar$ may be replaced by a rational number $\frac{M}{N}$, $(M, N) = 1$, for some "huge" integer $N$ with the property that every "feasible" integer $m$ divides $N$.

Note that $\hbar$ is measured in physical units "energy" $\times$ "time", so we have to formulate $\hbar$ in some absolute units before applying the above considerations. We have to make a physical assumption, which appears to be consistent with some thinking in modern physics (see e.g. p.92 [13], article by S.Majid)

**Physics assumption.** *In some natural physical units*

$$\hbar = \frac{1}{N}$$

*for some positive integer* $N$, *such that* $N$ *is divisible by any "feasible" integer* $m$.

Here, "feasible" could mean *small enough integer to be tested in physical experiments.*

We are going to discuss consequences of this assumption below, in particular formulas for the Feynman propagator for the free paricle and for the quantum harmonic oscillator. These formulas may become false if the corresponding integer parameters in the formulas exceed the treschold of the feasibility in the above sense.

# 5 Dirac calculus over the 2-torus

**5.1** Dirac calculus was initially a heuristic method of calculation of results of transformations of "wave functions" (corresponding to states of a physical system) by operators expressed in terms of $Q$ (position) and $P$ (momentum) satisfying the Heisenberg commutation relation (5). By effors of D.Hilbert, J.von Neumann and many other mathematicians this has been developed into a rigorous mathematical theory (of self-adjoint operators in "rigged" Hilbert spaces) which covers most (but not all!) needs of quantum mechanics. By this theory a wave function is just a particular element of the corresponding Hilbert space, $\mathcal{H}$, preferably, expressed in terms of the basis of normalised eigenvectors of the operator $Q$ (or Fourier-dually, of the operator $P$). The most important of such transformation is one by the **time evolution** operator depending on the time $t$,

$$K^t := e^{-it\frac{H}{\hbar}}, \tag{12}$$

where $H$ is the **Hamiltonian,** a self-adjoint operator of the form

$$H = \frac{P^2}{2\hbar} + V(Q),$$

$V(Q)$ a function of $Q$ (the potential), often a polynomial.

By specifying $V(Q)$ one determines the type of the physical particle under consideration, and the physics of the particle is fully determined by the corresponding time evolution operator (12). In concrete terms this boils down to evaluating certain integrals which often turn out to be non-convergent. So, again, physicists invent various heuristic methods of evaluation of the integrals justified partially by a mathematical analysis and partially by experimental tests.

**5.2** For Dirac calculus one uses the standard bases of the Hilbert space $\mathcal{H}$, the basis consisting of eigenvectors $|p\rangle$ for the operator $P$, where $|p\rangle$ denotes an eigenvector of norm 1 with an eigenvalue $p$, and the Fourier-dual basis $|x\rangle$ of $Q$-eigenvectors of norm 1, where again $x$ is the corresponding eigenvalue. This is in a clear analogy to the classical Hamiltonian system, a 2-space (in the simplest 1-dimensional case) with two coordinate function, the momentum $p$ and the position $x$.

Elements of the Hilbert space, which are eigenvectors of self-adjoint operators of physical meaning are usually called **states**. In this sense the Hilbert space $\mathcal{H}$ is the home of the "space of quantum states", the nearest analogue of the classical space where the Newtonian physics takes place. We extract from this arrangement a proper geometric structure - a Zariski geometry.

**5.3** Our first step is to replace the operators $P$ and $Q$ with eventually equivalent (Weyl) operators $V = \exp iaP$ and $U = \exp ibQ$, for some $a, b \in \mathbb{R}$. In particular, for $a = 1, b = 2\pi$ (5) implies, by the Baker-Campbell-Hausdorff formula, the Weyl commutation relation (1).

Note that a $P$-eigenvector $|p\rangle$ is by definition an eigenvector of $V$ with eigenvalue $\exp iap$, and similarly with eigenevectors of $Q$ and $U$.

*The whole structure of $V$- and $U$-eigenvectors is represented as the geometric quantum 2-torus as described in section 3. This is a Zariski structure.*

More generally denote $U^a = \exp 2\pi iaQ$ and $V^b = \exp ibQ$, for rational numbers $a$ and $b$ ($U^1 := U$, $V^1 := V$, by definition). This will satisfy the more general form of the commutation relation (1),

$$V^b U^a = q^{ab}\, U^a V^b, \quad q^{ab} = \exp 2ab\pi i\hbar. \tag{13}$$

The structure of $U^a$- and $V^b$-eigenvectors is by the same reasons as above represented as the geometric quantum 2-torus (or as the $(U^a, V^b)$-system) and is a Zariski structure. Moreover, for natural numbers $m, n$, an $(U^{\frac{1}{m}}, V^{\frac{1}{n}})$-system can be seen as an étale covering of the $(U, V)$-system and the latter is definable in the former. Altogether, the $(U^a, V^b)$-systems through the web of étale covers converge to a **$(Q, P)$-system**, a linear space with an action of operators $P$ and $Q$ on it, akin to the Hilbert space $\mathcal{H}$ above.

More precisely, there is a map

$$\begin{array}{l} \exp^{a,b} : |x\rangle \mapsto \mathbf{u}(e^{2\pi a x}, 1), \\ \exp^{a,b} : |p\rangle \mapsto \ \mathbf{v}(e^{bp},\ 1), \end{array} \tag{14}$$

from $\mathcal{H}$ to respective eigenvectors of $(U^a, V^b)$-systems, which allows to treat $|x\rangle$ and $|p\rangle$ as limits of $\mathbf{u}(e^{2\pi a x}, 1)$ and $\mathbf{v}(e^{bp}, 1)$ respectively as $a$ and $b$ converge to 0.

This passes often ill-posed Dirac calculus problems into the context of Zariski structures "at roots of unity".

**5.4** The next step is to include the time evolution operator into the Zariski geometric setting, that is to extend a $(U^a, V^b)$-system to an $(U^a, V^b, K^t)$-system, for $K^t$ the operator defined by (12) above, for a choice of $H$.

Consider first the simplest case, the **free particle** Hamiltonian,

$$H = \frac{\mathrm{P}^2}{2\hbar}.$$

We choose to work only with values of time $t$ such that $2\pi t$ is rational. This is quite satisfactory in terms of practical purposes of the physicist.

Now use the exponential correspondence between $Q, P$ and $U, V$ to calculate by the Baker-Campbell-Hausdorff formula (just like physicists do) that for the $K^t$ defined in (12) for the free particle Hamiltonian above,

$$K^t U K^{-t} = q^{-\pi t} U V^{-2\pi t}, \quad K^t V K^{-t} = V. \tag{15}$$

We define an $(U, V^{2\pi t}, K^t)$-algebra as an algebra generated by the three operators and their inverses and satisfying (13) (with $a = 1, b = 2\pi t$) and (15). Note, by using linear algebra, that in any irreducible finite-dimensional $(U, V^{2\pi t})$-module one can define an operator $K^t$ satisfying (15) uniquely, up to the multiplication by a normalising constant.

We define an $(U, V^{2\pi t}, K^t)$-system abstractly as an $(U, V^{2\pi t})$-system with an action of the $(U, V^{2\pi t}, K^t)$-algebra, extending the action of the $(U, V^{2\pi t})$-algebra.

*Under a certain choice of the normalising constant an $(U, V^{2\pi t}, K^t)$-system is determined uniquely, up to isomorphism over $\mathbb{C}$. (In particular, its first-order theory is categorical in uncountable cardinals). Moreover, we expect that the system is a Zariski geometry.*

**5.5** Now we work out the content of the previous subsection in more detail, assumin that $q$ is a root of unity of order $N$. We choose $t$ so that $2\pi t$ is a rational number (with a feasible numerator).

Let $\{\mathbf{u}(u,v) : u \in \mathbb{C}^*,\ v \in \Phi\}$ be the system of $U$-eigenvectors for the $(U, V^{2\pi t}, K^t)$-system, that is, by definition,

$$U : \mathbf{u}(u,v) \mapsto u\mathbf{u}(u,v), \quad V^{2\pi t} : \mathbf{u}(u,v) \mapsto v^{2\pi t}\mathbf{u}(u,v).$$

Note that the dimension of the module is $\frac{N}{2\pi t}$ which is an integer according to the assumption in subsection 4.7.

Denote

$$\mathbf{s}(u,v) := K^t\, \mathbf{u}(u,v), \quad S_t := q^{-\pi t} U V^{-2\pi t}.$$

Then, it is easy to derive from (15) that for all $u \in \mathbb{C}^*$, $v \in \Phi$, $k \in \mathbb{Z}$ :

$$S_t\, \mathbf{s}(u,v) = u\, \mathbf{s}(u,v), \quad V^{2\pi k t}\, \mathbf{s}(u,v) = v^{2\pi k t}\, \mathbf{s}(q^{-2\pi k t}u, v). \tag{16}$$

Using these formulas one gets

$$\begin{aligned} &\big(\mathbf{s}(u,v) \,|\, \mathbf{u}(uq^{2\pi k t}, v)\big) = c\, q^{\pi t k^2} v^{-2\pi k t}, \\ &\text{where } c = c(u, v^{2\pi t}) = q^{-\pi t} \left(\mathbf{s}(u,v) \mid \mathbf{u}(u,v)\right). \end{aligned} \tag{17}$$

In fact $c$ depends only on the corresponding $(U, V^{2\pi t}, K^t)$-module but not on the concrete choice of $u$ and $v$.

Moreover, for physically meaningful case, $u, v$ are of modulus 1 and then $U$ and $V^{2\pi t}$ are unitary operators on the inner product space. It follows

$$c = c_0 \sqrt{\frac{2\pi t}{N}}, \text{ for some } c_0 \text{ with } |c_0| = 1.$$

Finally, we rewrite (17) in the equivalent form

$$\mathbf{s}(u,v) = c_0 \sqrt{\frac{2\pi t}{N}} \sum_{k=0}^{\frac{N}{2\pi t}-1} q^{\pi t k^2} v^{-2\pi k t} \mathbf{u}(uq^{2\pi k t}, v) \tag{18}$$

### 5.6 Alternative representation and the value of $c_0$

Recall that in the canonical Dirac calculus $P : |p\rangle \mapsto p|p\rangle$ and so

$$e^{-it\frac{\mathrm{P}^2}{2\hbar}} : \ |p\rangle \mapsto e^{-\frac{itp^2}{2\hbar}}|p\rangle.$$

Since $e^{-ithp^2} = q^{-tp^2}$, in correspondence with (14) and (12)

$$K^t : \mathbf{v}(q^{2\pi pt}, 1) \mapsto q^{-\pi tp^2}\mathbf{v}(q^{2\pi pt}, 1),$$

Recall that in an irreducible $(U, V^{2\pi t})$-module

$$\mathbf{u}(u, 1) = \sum_{p=0}^{\frac{N}{2\pi t}-1} \mathbf{v}(q^{2\pi pt}, u).$$

Hence

$$K^t : \ \mathbf{u}(1, 1) \mapsto \sqrt{\frac{2\pi t}{N}} \sum_{p=0}^{\frac{N}{2\pi t}-1} q^{-\pi tp^2}\mathbf{v}(q^{2\pi pt}, 1).$$

This shows that the right-hand side of the above can be identified as $K^t\mathbf{u}(1, 1)$, that is

$$\mathbf{s}(1, 1) := \sqrt{\frac{2\pi t}{N}} \sum_{p=0}^{\frac{N}{2\pi t}-1} q^{-\pi tp^2}\mathbf{v}^{1,2\pi t}(q^{2\pi pt}, 1). \tag{19}$$

Substituting (10) (with corresponding parameters) into (19) we get

$$\mathbf{s}(1, 1) := \sqrt{\frac{2\pi t}{N}} \sum_{p=0}^{\frac{N}{2\pi t}-1} q^{-\pi tp^2}\sqrt{\frac{2\pi t}{N}} q^{2\pi pkt} \sum_{k=0}^{\frac{N}{2\pi t}-1} \mathbf{u}(q^{2\pi kt}, 1) =$$

$$= \frac{2\pi t}{N} \sum_{k=0}^{\frac{N}{2\pi t}-1} \sum_{p=0}^{\frac{N}{2\pi t}-1} q^{-\pi t(p^2-2pk)}\mathbf{u}(q^{2\pi kt}, 1).$$

Finally we compare the latter with (18) and equate the coefficients in front of $\mathbf{u}(q^{2\pi kt}, 1)$.

$$\frac{2\pi t}{N} \sum_{0\le p<\frac{N}{2\pi t}} q^{-\pi t(p^2-2pk)} = c_0\sqrt{\frac{2\pi t}{N}} q^{\pi tk^2}, \tag{20}$$

which can be rewritten, for $k=0$, $M := \frac{N}{2\pi t}$, as

$$\sum_{p=0}^{M-1} e^{-i\frac{\pi p^2}{M}} = c_0\sqrt{M}.$$

And here finally we can establish the value of the constant

$$c_0 = \frac{1-i}{\sqrt{2}},$$

as the above is the classical Gauss' sum (see [9]), for an even (!) integer $M$ :

$$\sum_{p=0}^{M-1} e^{-i\frac{\pi p^2}{M}} = \frac{1-i}{\sqrt{2}}\sqrt{M}.$$

**5.7 The kernel of the Feynman propagator.**

This is defined as

$$\langle x_1 | K^t | x_0 \rangle, \tag{21}$$

where $|x_0\rangle$ and $|x_1\rangle$ are position eigenvectors and the formula should be read as the inner product (pairing) of $K^t|x_0\rangle$ and $|x_1\rangle$.

Note that $e^{2\pi i x} = q^{\frac{x}{\hbar}}$. So in accordance with (11) and (14) we rewrite (21), assuming that $\frac{x_i}{\hbar}$ are rational, $i = 0,1$, as

$$\frac{N}{2\pi t} \cdot \Big( \mathbf{s}(q^{\frac{x_0}{\hbar}}, 1) \,|\, \mathbf{u}(q^{\frac{x_1}{\hbar}}, 1) \Big)$$

(note that $N$ in (11) should be now $\frac{N}{2\pi t}$, the dimension of the $(U, V^{2\pi t}, K^t)$-module). Now use (17) with $x_0 = 2\pi t m\hbar$, $u = q^{2\pi t m}$, $x_1 = 2\pi(m+k)t\hbar$, $uq^{2\pi k t} = q^{2\pi(m+k)t}$, to get

$$\Big( \mathbf{s}(q^{\frac{x_0}{\hbar}}, 1) \,|\, \mathbf{u}(q^{\frac{x_1}{\hbar}}, 1) \Big) = c_0 \sqrt{\frac{2\pi t}{N}} e^{i\frac{(x_1 - x_0)^2}{2t\hbar}}$$

some $c_0$ of modulus 1.

We will establish below that the only value for $c_0$ consistent with the meaning of $K^t$ as the time evolution operator is

$$c_0 = \frac{1-i}{\sqrt{2}} = \sqrt{-i} = \sqrt{\frac{1}{i}}.$$

Recall also that by our assumptions

$$\frac{1}{N} = \hbar.$$

This finally gives us the value for $K^t$, the kernel of the Feynman propagator for the free particle,

$$\langle x_1|K^t|x_0\rangle = \sqrt{\frac{1}{2\pi i\hbar t}}\, e^{i\frac{(x_1-x_0)^2}{2t\hbar}}$$

**5.8** The final expression is the well known formula, see e.g. [7], formula (7.76). The standard method of obtaining this result is to assign a meaning to the following Gaussian integral

$$I(a) = \int_{\mathbb{R}} e^{-ax^2/2} e^{-ipx} dx$$

for $a = \frac{i}{t\hbar}$. This integral is divergent for imaginary values of $a$ but is convergent for $a$ on the right halfplane. Physicists assume the value of $I(\frac{i}{t\hbar})$ to be determined by the analytic continuation of $I(z)$ from the right halfplane to its boundaries. Note that similarly physicists use the trick with analytic continuation to give finite meanings to infinitary expressions, e.g. the formula

$$\sum_{n=1}^{\infty} n^3 = \zeta(-3) = \frac{1}{120}$$

is used to calculate the so called *Casimir effect* which, remarkably, is testable by an experiment.

**5.9** The same scheme is applicable in the case of the **simple Harmonic oscillator** given by the Hamiltonian

$$H = \frac{\mathrm{P}^2 + \omega^2\mathrm{Q}^2}{2\hbar}.$$

We choose the parameter $\omega$ (angular frequency) to be $2\pi$, which is a natural unit of measure here.

First we obtain an analogue of the formulas (15) for $K^t$ corresponding to the Harmonic oscillator:

$$K^tUK^{-t} = q^{-\frac{1}{2}\sin 2\pi t\cos 2\pi t}U^{\cos(-2\pi t)}V^{\sin(-2\pi t)} =$$
$$= q^{\frac{1}{2}\sin 2\pi t\cos 2\pi t}V^{\sin(-2\pi t)}U^{\cos(-2\pi t)}; \qquad (22)$$

$$K^tVK^{-t} = q^{\frac{1}{2}\sin 2\pi t\cos 2\pi t}U^{\sin 2\pi t}V^{\cos 2\pi t} =$$
$$= q^{-\frac{1}{2}\sin 2\pi t\cos 2\pi t}V^{\cos 2\pi t}U^{\sin 2\pi t}. \qquad (23)$$

This is a result of calculations in the Banach algebra generated by operators $P$ and $Q$ and is well-known to physicists.

Next, we fix a value for $t$ such that both $\sin 2\pi t$ and $\cos 2\pi t$ are nonzero rational numbers. Note that there are infinitely many such because the equation $x^2 + y^2 = 1$ determines a rational curve.

We note that the operators $U$ and $V^{\sin(-2\pi t)}U^{\cos 2\pi t}$ form a pair of Weyl operators and we work with irreducible modules of the $(U, V^{\sin(-2\pi t)}U^{\cos(-2\pi t)}, K^t)$-algebra defined by relations (1),(22) and (23). It is easy to see that the dimension of the irreducible modules is $\frac{N}{|\sin 2\pi t|}$ which is an integer by our assumptions in 4.7.

Now we are able to calculate $K^t\mathbf{u}(u,1)$ in terms of the basis of $U$-eigenvectors, uniquely up to a constant $c_0$ of modulus 1. This finally produces, similarly to 5.7, the formula for the Feynman propagator for a simple Harmonic oscillator,

$$\langle x_2|K^t|x_1\rangle = c_0\sqrt{\frac{1}{\hbar|\sin 2\pi t|}}\,\exp \pi i\frac{(x_1^2 + x_2^2)\cos 2\pi t - 2x_1x_2}{\hbar\sin 2\pi t}.$$

In accordance with the well-known formula [8], p552, for $\omega = 2\pi$.

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