Model Theory of Analytic and Pseudo-analytic Structures

Report on EPSRC project

Summary of outcomes

Counterparts of some classical analytic structures, e.g. the complex numbers with exponentiation, has been constructed and proven to satisfy the crucial model theoretic properties: excellence and categoricity. It has been shown that the abstract Shelah's model theory of these structures is closely connected with Schanuel's conjecture (transcendental number theory), Mordell-Lang type (Diophantine) conjectures and the Kummer theory for semi-abelian varieties. New type of analyticity, combining the complex and the real structures has been discovered.

New results in the abstract theory of excellent classes, related to the examples above, have been obtained.

Outcomes in more detail

Much of the work under the project has been aimed towards understanding the links between basic model theoretic (stability) concepts and classical analyticity. It has been suggested by the principal investigator [Zilber6] that the only known construction of 'new stable structures', the Hrushovski construction, can be explained in terms of complex analytic functions or maybe 'analytic' in a more general sense. In particular, one of the main ingredients of Hrushovski's construction, the predimension inequality, thus justifies Schanuel-type conjectures for classical transcendental analytic functions. The paper [Wilkie5] provides an example of a class of complex analytic (Liouville) functions which supports this vision, in particular the constructed functions satisfy a desired 'Schanuel's condition'. Later P.Koiran proved other desired properties of Liouville functions thus establishing the fact that Wilkie's structures are elementarily equivalent to the ones obtained by a certain Hrushovski construction. In [Zilber2] it is shown, using special case of [Zilber9], that indeed the Hrushovski construction with a certain input produces structures isomorphic to Wilkie's structures on the complex field.

More involved and mathematically charged is the case of modelling the behavior of the complex exponentiation. Here every ingredient of the previous example becomes highly non-trivial. In the first part of [Zilber8] using a version of Hrushovski's construction, an expansion of an algebraically closed field by a function ex is constructed, with Hrushovski's predimension inequality taking exactly the form of Schanuel's conjecture. It is then shown that the class of such *fields with pseudo-exponentiation* can be axiomatised by a $L_{\omega_1,\omega}(Q)$-sentence. We argue in [Zilber6] that to show that structures constructed in this way are *canonical,* one needs to prove that the sentence determines essentially a unique structure in every cardinal. That is, the class is categorical in uncountable cardinals. Here is where Shelah's theory of categoricity steps in. In the spirit of this theory we prove in [Zilber9] a categoricity theorem which provides a test applicable for quasiminimal classes, including the one in question. But checking the assumptions of the test amounts to checking quite involved arithmetic statements involving Galois and Kummer theory. This number-theoretic work has been done in [Zilber5]. Using this result the categoricity of the class of fields with pseudo-exponentiation has been proved.

A similar, but mainly first-order, approach has been used to study a simpler class of structures, modelling the behavior of the complex numbers with the multivalued operations $x \mapsto x^a$, for $a \in \mathbb{C}$. Under a generalisation of Mordell-Lang conjecture [Zilber4] proves that this theory is superstable and is near model complete. Recently in [Zilber12] the same result has been proved without any assumption.

Two lines of research lead from here. One is to explore the connection between the arithmetic and model theoretic properties, the second is to ask which of the model theoretic properties established for pseudo-exponentiation do hold for actual exponentiation. The second task is undertaken in [Zilber1] where it is shown that under Shanuel's conjecture the actual operation $x \mapsto x^a$, for $a \in \mathbb{R}$ on $\mathbb{C}$ has superstable and near model complete theory, and an application of this to solving systems of exponential equations is given. Recently Wilkie has proved that for generic values of $a \in \mathbb{R}$ the correspondent fragment of Schanuel's conjecture does hold, so the theorem holds unconditionally (in preparation).

The connection to arithmetic is studied in [Zilber3]. It has been proved that the analogue of the categoricity statement for pseudo-exponentiation for semi-abelian varieties is equivalent to a conjunction of very strong arithmetic statements about semi-abelian varieties. Some of the statements, like Mordell's theorem about rational points of abelian varieties, are well-known. Some others are known as conjectures and some can be seen as new arithmetic conjectures, supported by the model theory. Further essential progress in this direction has been achieved in the work of M.Gavrilovich in his graduate research at Oxford (DPhil thesis *Model Theory of the Universal Cov-*

ering Spaces of Complex Algebraic Varieties submitted October 2005, see http://misha.uploads.net.ru).

Another important aspect of the work is the development of abstract model-theoretic tools relevant to the study of analytic and pseudo-analytic structures. The paper [Lessmann1] is written to a large extent for this particular purpose. It puts the existing theory on a more adequate, for application purposes, footing. In particular the paper explores the notion of quasiminimal excellence. [Lessmann3] makes further progress in this direction developing the non-elementary categoricity theory along the Baldwin-Lachlan line. He shows that, analogously to the first order case, if an excellent class is categorical in some uncountable cardinal, each model is prime and minimal over the basis for the pregeometry on a quasiminimal set.

The above mentioned results, [Zilber 6] [Zilber3] and the work of Gavrilovich demonstrated that the arithmetic necessary for establishing $\aleph_1$-categoricity of the pseudo-analytic classes is considerably easier than the one needed to establish categoricity in all uncountable cardinals. This naturally leads to the question of exploring the passage from $\aleph_1$-categoricity to full uncountable categoricity for relevant types of theories. To this end Grossberg and Van-Dieren introduced the notion of a *tame abstract elementary class*. [Lessmann4] based on Grossberg - Van-Dieren proves the very important result that, for a tame abstract elementary class, $\aleph_1$-categoricity implies full uncountable categoricity. Baldwin in the forthcoming book [Baldwin] discusses this and other issues in connection with the pseudo-analytic classes studied in our project.

A very essential issue of the project is the interaction between stability and o-minimality, as model theoretic technologies. The first, as is well-known, has algebraically closed fields as the archetype, while the latter is based on the reals. The technique used by Wilkie in the results mentioned above combines them both. Importantly, the new example [Wilkie6] answering a well-known question of Steinhorn on o-minimal structure on the rational numbers is obtained by a method resembling the Hrushovski construction (designed originally to produce new stable structures).

The work of project student G.Jones concentrates on developing cohomology theory for abstract o-minimal structures. In [Jones1] it is proved that in any o-minimal expansion of a group there exist cohomology groups $H^i(X, \mathbf{Z})$, for all $i \geq 0$, for any definable subsets X.

In [Zilber7] it is shown that the known difficult example of Poizat of a *bad field* (a Hrushovski-type structure) can be represented as a pseudo-analytic structure. Very importantly, in this interpretation and the corresponding proofs, the complex and the real analytic structures interplay in a very specific way. We consider it a new type of analyticity to be studied; speculations connecting this and other examples to non-commutative geometry are available in [Zilber13]. Finally, the joint work of Peatfield and Zilber [Zilber10] introduces the formalism of an *analytic Zariski structure*. We consider it an important step in developing an abstract theory of analyticity and expect most (if not all) of the above mentioned pseudo-analytic and analytic structures to satisfy this definition. The lecture course [Zilber14] discusses this and other issues in more detail.

Publications of the participants of the project

Lessmann:

1.An introduction to excellent classes. Logic and its applications, 231-259, Contemp. Math., 380, Amer. Math. Soc., Providence, RI, 2005.

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3.Categoricity and U-rank in excellent classes. J. Symbolic Logic 68 (2003), no. 4, 1317-1336.

4.Upward categoricity from a successor cardinal for tame abstract classes with amalgamation. J. Symbolic Logic 70 (2005), no. 2, 639-660.

5.(with Hyttinen and Shelah) Interpreting groups and fields in some nonelementary classes. J. Math. Log. 5 (2005), no. 1, 1-47.

Jones: 1.(with Peatfield and Edmundo) Sheaf cohomology in 0-minimal structures. Newton Institute preprint NI05029-MAA, also submitted to the Journal of Mathematical Logic.

2. (with Peatfield and Edmundo) Hurewicz theorems for definable groups. Preprint.

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2.The laws of integer divisibility,and solution sets of linear divisibility conditions (with L van den Dries),J Symbolic Logic,68 (2),2003,503-526.

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5. Covers of the multiplicative group of an algebraically closed field of characteristic zero, E-archive: math.AC/0401301; accepted in J.London Math.Soc.

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9. A categoricity theorem for quasiminimal excellent classes. In Logic and its Applications eds. Blass and Zhang, Cont.Maths, v.380, 2005, pp.297-306

10.(with N.Peatfield) Analytic Zariski structures and the Hrushovski construction. Annals of Pure and Applied Logic, Vol 132/2-3 (2005) pp 127-180

11. (with J.Kirby) The uniform Schanuel conjecture over the real numbers. To appear in the Bulletin of LMS

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