Poizat's bad fields and quantum groups

Boris Zilber

University of Oxford

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http://www.maths.ox.ac.uk/∼zilber

Bad fields

Definition A bad field is a structure $(K, +, \cdot, G)$ with $\mathrm{MR}(K) = N > 1$ and $G < K^\times$, a multiplicative subgroup, $\mathrm{MR}(G) = 1$.

Problem (197?) Do bad field exist?

Theorem (Baldwin and Holland, 2002) Yes, for each $N > 1$, if we drop the requirement that G is a group.

Theorem (B.Poizat, 2000) There exist an almost bad fields $(K, +, \cdot, G)$ with $\mathrm{MR}(K) = \omega \times N$ and $G < K^\times$, a multiplicative subgroup, $\mathrm{MR}(G) = \omega$.

Proof. Based on Hrushovski's construction.

Easy case: $G = P$ is just a subset (of black points).

Predimension: $\delta(X) = N \cdot \text{tr.deg}(X) - \text{size}(X \cap P)$.

Axioms:
GSCH: For distinct $x_1, \ldots, x_k \in P$,

$$N \cdot \text{tr.deg}(x_1, \ldots, x_k) - k \geq 0.$$

EC: Let $V \subseteq \mathbb{C}^k$ be an irreducible algebraic variety defined over a finitely generated subfield $\mathbb{Q}(C)$ which has a point
$\langle a_1, \ldots, a_k \rangle \in V$ satisfying:
$a_i \neq a_j$ and $a_i \notin \text{acl}(C)$, $i \neq j, i \leq k$ **(P-free)**
and
$\text{tr.deg}(a_{i_1}, \ldots, a_{i_m}) \geq \frac{m}{N}$, for any $i_1 < \cdots < i_m \leq k$ (P-**normal**).

Then there is $\langle a_1, \ldots, a_k \rangle \in V \cap P^k$.

Claim. GSCH and EC for $(K, +, \cdot, P)$ are first order.
Proof. Easy $\Box$

Difficult case: G is a subgroup of the multiplicative group (green points):
Predimension

$$\delta(X) = N \cdot \text{tr.deg}(X) - \text{mult.rk}(X \cap G).$$

Axioms:

GSCH: For multiplicatively independent $x_1, \ldots, x_k \in G$,

$$N \cdot \text{tr.deg}(x_1, \ldots, x_k) - k \geq 0.$$

EC: Let $V \subseteq \mathbb{C}^k$ be an irreducible algebraic variety defined over a finitely generated subfield $\mathbb{Q}(C)$ and which is G-free and G-normal.
Then there is $\langle a_1, \ldots, a_k \rangle \in V \cap G^k$.

Claim. GSCH and EC for $(K, +, \cdot, G)$ are first order.
Proof. Uses Ax's Theorem on Schanuel's conjecture for function fields. $\square$

Problem. Explain these examples analytically.

Solutions, for $N = 2$.

$K = \mathbb{C}$.

Case: P is a "generic" subset of $\mathbb{C}$.

Let ϵ, α be algebraic numbers, $\epsilon \notin \mathbb{R} \cup i\mathbb{R}$, $\alpha \notin \mathbb{Q}$, $\alpha\mathbb{R} \neq \epsilon\mathbb{R}$.
Let f be a Liouville function (A.Wilkie) and

$$P = \{f(\epsilon t + \alpha q) : t \in \mathbb{R},\ q \in \mathbb{Q}\}$$

Theorem $(\mathbb{C}, +, \cdot, P)$ is a model of Poizat's "black points" theory.

Case: G is a multiplicative subgroup of $\mathbb{C}$.

Let

$$G = \{\exp(\epsilon t + \alpha q) : t \in \mathbb{R},\ q \in \mathbb{Q}\}$$

Theorem. Assume Schanuel's conjecture. Then $(\mathbb{C}, +, \cdot, G)$ is a model of Poizat's "green points" theory.

Problem 1. Do the general case N.

Remark Necessarily, the real dimension of the bad subset (P or G) must be $\frac{2}{N}$.
Thus, the theory of fractional dimensions needs to be involved in the construction.

Let $h \in \mathbb{R} \setminus \mathbb{Q}$, $\epsilon \in (\mathbb{C} \setminus \mathbb{R} \cup i\mathbb{R})$,

$$G = \{\exp(\epsilon t + 2\pi i h m) : t \in \mathbb{R},\ m \in \mathbb{Z}\}$$

$$\Gamma = \{\exp(2\pi i h m) :\ m \in \mathbb{Z}\}$$

Theorem Assume Schanuel's conjecture. Then

1.Th$(\mathbb{C}, +, \cdot, G,\ \Gamma)$ is superstable,

$$\mathrm{U}(\mathbb{C}) = \omega \cdot 2,\ \mathrm{U}(G) = \omega,\ \mathrm{U}(\Gamma) = 1.$$

2. The Miller-Speissegger spiral $G^0 = \exp(\epsilon\mathbb{R})$ is type-definable in $(\mathbb{C}, +, \cdot, G)$ as the *connected component* of G.

3. The field of reals is $L_{\omega_1,\omega}$-definable in $(\mathbb{C}, +, \cdot, G)$.

Problems

2.Describe *canonical* models in the elementary class $\mathrm{Th}(\mathbb{C}, +, \cdot, G,\ \Gamma)$.

3.Put $(\mathbb{C}, +, \cdot, G,\ \Gamma)$ in the context of *analytic Zariski* structures.

The quantum torus

Theorem Given an algebraically closed field F and its cyclic multiplicative subgroup $\Gamma = \langle q \rangle$,
the two-sorted structure

$$(\mathrm{F}, +, \cdot) \xrightarrow{\theta} (T, \cdot)$$

$$(\mathrm{Dom}\ \theta = \mathrm{F}^*,\ \theta \text{ a homomorphism}, \ker\theta = \Gamma)$$

has an analytic Zariski structure on both sorts F and T.

Consider the F-vector space $\mathcal{H} = \mathcal{H}(T)$ of *local functions on infinitesimal neighborhood* $\mathcal{V}$ *of* $1 \in T$:

$$\mathcal{H}(T) = \{\psi : \mathcal{V} \subseteq {}^*T \to {}^*\mathrm{F}\}.$$

Example The inverse $x = \theta^{-1}$ to the map $\theta : {}^*\mathrm{F} \to {}^*T$ is well-defined on $\mathcal{V}$.
Consequently, for every $k \in \mathbb{Z}$

$$x^k : \ {}^*T \to {}^*\mathrm{F}$$

is well-defined. So, $x^k \in \mathcal{H}(T)$, $k \in \mathbb{Z}$.

There is an algebra $A(T)$ of *definable* linear operators acting on $\mathcal{H}(T)$:

$$U : \ x^k \to x^{k+1},$$
$$V : \ x^k \to q^k x^k.$$

$$A(T) := \langle V, U : \ VU = qUV \rangle.$$

Theorem Assuming Schanuel's conjecture the 3-sorted structure below is superstable and is known to have some analytic Zariski properties

$$\begin{array}{ccc} (\mathbb{C},+,h\cdot) & \xrightarrow{\exp v} & (\mathbb{C},+,\cdot) \\ \Big\downarrow \exp h^{-1}v & & \Big\downarrow \quad \theta \\ (\mathbb{C},+,\cdot) & \longrightarrow & T=\mathbb{C}^*/\Gamma \end{array}$$

$$\Gamma = \langle q \rangle, \quad q = \exp(2\pi i h).$$

In this language $\mathcal{H}(T)$ contains also the well-defined local functions

$$x^{kh} : \ {}^*T \to {}^*\mathrm{F}$$

with the action of the operators

$$U : \ x^{kh} \to q^k x^{kh},$$
$$V : \ x^{kh} \to x^{(k+1)h}.$$

Using the obvious symmetry between U and V we obtain the formal correspondence between the eigenvectors of the operators U and V :

$$x^{kh} \sim \sum_{m \in \mathbb{Z}} q^{-km} x^m.$$

Problem 4. Give a meaning to this formula.

Problem 5. Add the bad subgroup

$$G = \exp(\epsilon\mathbb{R} + 2\pi i h\mathbb{Z})$$

to the language and include the reals into the picture.

The quantum $\mathrm{SL}_q(2,\mathbb{C})$

Consider the action of the group

$$\mathbb{Z}\times\mathbb{Z}\cong\Gamma\times\Gamma=\{(q^m,q^n):\ m,n\in\mathbb{Z}\}$$

on $\mathrm{SL}(2,\mathbb{C})$:

$$\begin{pmatrix} X & Y \\ Z & V \end{pmatrix} \xrightarrow{(m,n)} \begin{pmatrix} Xq^n & Y \\ Zq^m & \frac{XV+YZ(1-q^m)}{Xq^n} \end{pmatrix}$$

This gives rise to the space of orbits

$$\mathrm{SL}_q(2,\mathbb{C})=\Gamma\times\Gamma\backslash\mathrm{SL}(2,\mathbb{C}).$$

Theorem The two-sorted structure

$$\mathrm{SL}(2,\mathbb{C})\xrightarrow{\theta}\mathrm{SL}_q(2,\mathbb{C})$$

is superstable and analytic Zariski in both sorts.

The above mentioned method of constructing an algebra A of linear operators acting on the space $\mathcal{H}$ of local functions

$$^*\mathrm{SL}_q(2,\mathbb{C}) \to^* \mathrm{SL}(2,\mathbb{C})$$

produces the $\mathbb{C}$-algebra with generators a, b, c, d and defining relations

$$\begin{gathered} ab = qba \\ bd = qdb \\ ac = qca \\ cd = qdc \\ bc = cb \\ ad - da = (q - q^{-1})bc \\ ad - qbc = da - q^{-1}bc = 1. \end{gathered}$$

This can be naturally made a *Hopf algebra* (with a comultiplication and a counit). This Hopf algebra

$$\mathcal{O}_q(\mathrm{SL}_2(\mathbb{C}))$$

is by definition the (algebraic) quantum $\mathrm{SL}(2,\mathbb{C})$.

Problem 6. Consider $\mathrm{SL}_q(2, \mathbb{C})$ in an expanded language involving the reals (as in Problem 5).

Problem 7. Study the model theory of the quantum unitary group $\mathrm{U}_q(2, \mathbb{C})$ and the quantum orthogonal group $\mathrm{O}_q(3)$.
Look for 'bad' stable groups related to these structures.