On Zariski geometries, structural approximation and rigorous Dirac calculus

B. Zilber

University of Oxford

http://www.maths.ox.ac.uk/~zilber/

Plan

I. Logical hierarchy of structures and Zariski geometries.

II. Zariski geometries as noncommutative spaces.

III. Zariski geometry for quantum Heisenberg relation.

IV. Structural approximation.

V. Time evolution and Feynman propagator for some Hamiltonians.

First question: in what sense the (mathematical) physics we study reflects the "real universe"? What is "real universe" for the mathematician?

First question: in what sense the (mathematical) physics we study reflects the "real universe"? What is "real universe" for the mathematician?

A plausible answer to the second part of the question: the real universe is a structure, say $\mathbf{M}$, that is a domain (set of points, "events", "particles",...) with some relations $R(x_1, ..., x_n)$ between its elements. The relations have some topological meaning:

$$R(\bar{x}) \Leftrightarrow r(\bar{x}) = 0$$

or

$$R(\bar{x}) \Leftrightarrow r(\bar{x}) \leq a$$

some nice function r. Sets defined by such relations are called closed.

Second question: what property of the "real" **M** allows laws of physics? why do we hope that a few laws of physics can describe **M**?

Philosophers call this property **"algorithmic compressibility"**.

In model theory we have a corresponding notion **categoricity in uncountable powers**: very large structure **M** describable uniquely by its (countable) first order theory.

Categoricity of $\mathbf{M}$ has strong structural consequences.

In combination with the topological assumption on $\mathbf{M}$ we come to the definition of *Zariski Geometry*:

Zariski structures

Let M be a structure given with a family of basic relations (subsets of M^n) called **closed**.

We postulate for a Noetherian Zariski structure M:

Closed subsets form a **Noetherian Topology**

Dimension is assigned to any closed $S \subseteq M^n$

Completeness: Projections of closed are closed

Addition formula:

$$\dim S = \dim \mathrm{pr}(S) + \min_{a \in \mathrm{pr}(S)} \dim(\mathrm{pr}^{-1}(a) \cap S)$$

for any closed irreducible S.

Pre-smoothness: For any closed irreducible $S_1, S_2 \subseteq M^n$,

$$\dim S_1 \cap S_2 \geq \dim S_1 + \dim S_2 - \dim M^n$$

in each component.

Known Noetherian Zariski structures

1. Smooth complete algebraic varieties over an algebraically closed field, in the natural language (1990).

2. Compact complex manifolds, in the natural language (1993).

3. Solution spaces of well-defined systems of (partial) differential equations. (2001)

4. Many non-commutative geometries (2003-...).

Classification Theorem (Hrushovski, Z. 199
and its later (Z. 2003-...) extensions.

A typical Zariski geometry is a "space of states" corresponding to a non-commutative C^*-algebra.

In particular, given a quantum algebra $\mathcal{A}$ at roots of unity there is a canonical construction of a Noetherian Zariski geometry $\mathbf{M} = \mathbf{M}(\mathcal{A})$

$$\mathcal{A} \quad \leftrightarrow \quad \mathbf{M}.$$

Classification Theorem (Hrushovski, Z. 1993) and its later (Z. 2003-...) extensions.

A typical Zariski geometry is a "space of states" corresponding to a non-commutative C^*-algebra.

In particular, given a quantum algebra $\mathcal{A}$ at roots of unity there is a canonical construction of a Noetherian Zariski geometry $\mathbf{M} = \mathbf{M}(\mathcal{A})$

$$\mathcal{A} \quad \leftrightarrow \quad \mathbf{M}.$$

In general *root of unity*, is not a necessary condition but some condition (*compactifiability?*) must be satisfied.

Example. The Heisenberg algebra (P, Q) with defining relation

$$\mathrm{QP} - \mathrm{PQ} = ih$$

is not compactifiable in this sense.

Example. The algebra (P, Q) with defining relation

$$\mathrm{QP} - \mathrm{PQ} = ih$$

is not compactifiable in this sense.

Let $U = e^{i\mathrm{Q}}$, $V = e^{i\mathrm{P}}$, $q = e^{ih}$. Then

$$UV = qVU$$

and $\mathcal{A}_q(\mathbb{C}) = (U, V)$ satisfies the compactness condition. Correspondingly,
there exists a 2*-dimensional Zariski structure* $T_q^2(\mathbb{C}$

$$T_q^2(\mathbb{C}) \quad \leftrightarrow \quad \mathcal{A}_q(\mathbb{C}).$$

$T_q^2(\mathbb{C})$ is a structure over $\mathbb{C}^* \times \mathbb{C}^*$ of two line bundles with connections:

U-bundle $|u, v\rangle$ with connection given by V,

$$U : |u, v\rangle \mapsto u \cdot |u, v\rangle \quad V : |u, v\rangle \mapsto v \cdot |uq, v\rangle$$

($|u, v\rangle$ is the trivialisation)

V-bundle with connection given by U,

$$V : |v, u\rangle \mapsto v \cdot |v, u\rangle \quad U : |v, u\rangle \mapsto u \cdot |vq^{-1}, u\rangle$$

($|v, u\rangle$ is the trivialisation)

Think of $|u, v\rangle$ as $|x\rangle$ and $|v, u\rangle$ as $|p\rangle$,
$u = e^{ix}$, $v = e^{ip}$.

$T_q^2(\mathbb{C})$ is a structure over $\mathbb{C}^* \times \mathbb{C}^*$ of two line bundles with connections:

U-bundle $|u, v\rangle$ with connection given by V,

$$U : |u, v\rangle \mapsto u \cdot |u, v\rangle \qquad V : |u, v\rangle \mapsto v \cdot |uq, v\rangle$$

($|u, v\rangle$ is the trivialisation)

V-bundle with connection given by U,

$$V : |v, u\rangle \mapsto v \cdot |v, u\rangle \qquad U : |v, u\rangle \mapsto u \cdot |vq^{-1}, u\rangle$$

($|v, u\rangle$ is the trivialisation)

Think of $|u, v\rangle$ as $|x\rangle$ and $|v, u\rangle$ as $|p\rangle$,
$u = e^{ix}$, $v = e^{ip}$.
Crucially, there is a formal *pairing*

$$\langle v, u | v', u' \rangle$$

The pairing can be properly explained only by approximating $T_q^2(\mathbb{C})$ with $T_\epsilon^2(\mathbb{C})$, $\epsilon -$ root of unity.

This corresponds to choosing

$$h = \frac{2\pi}{N}, \quad \text{positive integer } N.$$

The pairing can be properly explained only by approximating $T_q^2(\mathbb{C})$ with $T_\epsilon^2(\mathbb{C})$, ϵ − root of unity.
This corresponds to choosing

$$h = \frac{2\pi}{N}, \quad \text{positive integer } N.$$

Structural approximation. Let $\mathbf{M}_N$, $N \in I$, be a sequence of topological (Zariski) structures a compact structure $\mathbf{M}$ and an ultrafilter D on I such that there is a surjective homomorphism (preserves all "equations")

$$\prod_N \mathbf{M}_N/D \to \mathbf{M}.$$

We say in this case that the sequence of structures $\mathbf{M}_N$ approximates $\mathbf{M}$ along the ultrafilter D.

Structural approximation generalises algebro-geometr
deformation and metric approximation, e.g. *Gromov*
Hausdorff limit of metric spaces

Theorem (tentative). For a sequence $T^2_\epsilon(\mathbb{C})$ to approximate $T^2_q(\mathbb{C})$ it is sufficient and necessary that,
(a)

$$\lim_N \frac{2\pi}{N} = h$$

and
(b) given $m \in \mathbb{N}$, for almost all N (with respect to the ultrafilter) $m|N$.

Theorem (tentative). For a sequence $T^2_\epsilon(\mathbb{C})$ to approximate $T^2_q(\mathbb{C})$ it is sufficient and necessary that,
(a)

$$\lim_N \frac{2\pi}{N} = h$$

and
(b) given $m \in \mathbb{N}$, for almost all N (with respect to the ultrafilter) $m|N$.

Corollary. We may replace h by $\frac{2\pi}{N}$ such that $m|N$ for all $m << N$.

Now we work in an irreducible (U, V)-module generated by $|u, v\rangle$.

$$\{|uq^k, v\rangle \quad : k = 0, 1 \ldots N-1\}$$

forms an *orthonormal* basis of U-eigenvectors.

$$\{|vq^m, v\rangle \quad : m = 0, 1 \ldots N-1\}$$

a dual basis.

Now we work in an irreducible (U, V)-module generated by $|u, v\rangle$.

$$\{|uq^k, v\rangle \quad : k = 0, 1 \ldots N-1\}$$

forms an *orthonormal* basis of U-eigenvectors.

$$\{|vq^m, v\rangle \quad : m = 0, 1 \ldots N-1\}$$

a dual basis of V-eigenvectors.

$$|vq^m, u\rangle = \frac{1}{\sqrt{N}} \sum_{0 \le k < N} q^{-mk} |uq^k, v\rangle$$

$$|uq^k, v\rangle = \frac{1}{\sqrt{N}} \sum_{0 \le m < N} q^{km} |vq^m, u\rangle$$

$$N = \dim = \frac{2\pi}{h}.$$

What changes if we replace U, V by U^a, V^b, $a, b \in \mathbb{Q}$?
Then
$$U^a V^b = q^{ab} V^b U^a$$
and the dimension N of the irreducible module change
From the condition on structural approximation the new
$$\dim = \frac{N}{ab}.$$
Correspondingly

$$|vq^{abm}, u\rangle = \sqrt{\frac{ab}{N}} \sum_k q^{-abmk} |uq^{abk}, v\rangle$$
$$|uq^{abk}, v\rangle = \sqrt{\frac{ab}{N}} \sum_m q^{abkm} |vq^{abm}, u\rangle$$

Time evolution for the free particle

Choose $t = \frac{m}{n}$ and add one more operator

$$K^t = e^{i\frac{\mathrm{P}^2}{2h}t}.$$

Then

$$K^t V K^{-t} = V \text{ and } K^t U K^{-t} = q^{\frac{t}{2}} V^t U.$$

We take these identities for an axiomatic definition of K^t.

Time evolution for the free particle

Choose $t = \frac{m}{n}$ and add one more operator

$$K^t = e^{i\frac{\mathrm{P}^2}{2h}t}.$$

Then

$$K^t V K^{-t} = V \text{ and } K^t U K^{-t} = q^{\frac{t}{2}} V^t U := S.$$

We take these identities for an axiomatic definition of K^t.

Lemma K^t maps the (orthonormal) system $|u, v\rangle$ of U-eigenvectors to an orthonormal system of S-eigenvectors $|u, v\rangle_S$.

Time evolution for the free particle

Choose $t = \frac{m}{n}$ and add one more operator

$$K^t = e^{i\frac{\mathrm{P}^2}{2h}t}.$$

Then

$$K^t V K^{-t} = V \text{ and } K^t U K^{-t} = q^{\frac{t}{2}} V^t U := S.$$

We take these identities for an axiomatic definition of K^t.

Lemma K^t maps the (orthonormal) system $|u, v\rangle$ of U-eigenvectors to an orthonormal system of S-eigenvectors $|u, v\rangle_S$.

$$|u, 1\rangle_S = c_0 \sqrt{\frac{t}{N}} \sum_{0 \leq k < \frac{N}{t}} q^{t\frac{k^2}{2}} |uq^{-tk}, 1\rangle$$

$$|c_0| = 1.$$

$$|u,1\rangle_S = c_0\sqrt{\frac{t}{N}}\sum_{0\le k<\frac{N}{t}} q^{t\frac{k^2}{2}}|uq^{-tk},1\rangle$$

$$|c_0| = 1.$$

Corollary.

$$\langle x_1|K^t|x_2\rangle = c_0\sqrt{\frac{ht}{2\pi}}e^{i\frac{(x_1-x_2)^2}{2ht}}.$$

$$|u,1\rangle_S = c_0\sqrt{\frac{t}{N}} \sum_{0\le k<\frac{N}{t}} q^{t\frac{k^2}{2}} |uq^{-tk},1\rangle$$

$$|c_0| = 1.$$

Corollary.

$$\langle x_1|K^t|x_2\rangle = c_0\sqrt{\frac{ht}{2\pi}} e^{i\frac{(x_1-x_2)^2}{2ht}}.$$

Alternatively we can express

$$|u,1\rangle_S$$

in terms of $|vq^m,u\rangle$ and then use

$$|vq^m,u\rangle = \frac{1}{\sqrt{N}} \sum_{0\le k<N} q^{-mk} |uq^k,v\rangle.$$

Thus we get another expression

$$|u,1\rangle_S = \frac{t}{N} \sum_{k<\frac{N}{t}} \sum_{p<\frac{N}{t}} q^{-t(\frac{p^2}{2}-pk)} |uq^{kt},1\rangle.$$

Comparing the two expressions for $\langle x_1|K^t|x_2\rangle$, we get Gauss' sum

$$\sum_{0\leq p<\frac{N}{t}} q^{-t(\frac{p^2}{2}-pk)} = c_0\sqrt{\frac{N}{t}}q^{t\frac{k^2}{2}} = c_0\sqrt{\frac{2\pi}{ht}}e^{i\frac{(x_1-x_2)^2}{2ht}}$$

known to hold for even integers (!) $\frac{N}{t}$.

$$c_0 = \frac{1+i}{\sqrt{2}}$$

This corresponds to the usual (non-convergent) integral calculation

$$\frac{1}{\sqrt{2\pi}}\int e^{-ax^2/2}e^{-ipx}dx = \frac{1}{\sqrt{a}}e^{-p^2/2a}.$$

Time evolution for Harmonic oscillator

$$K^t = e^{it\frac{H}{h}}.$$

Similar strategy can be applied.
First we calculate in the usual way that

$$K_H^t U K_H^{-t} = q^{\frac{\sin t \cos t}{2}} V^{\sin t} U^{\cos t}$$

$$K_H^t V K_H^{-t} = q^{-\frac{\sin t \cos t}{2}} V^{\cos t} U^{-\sin t}$$

Time evolution for Harmonic oscillator

$$K^t = e^{it\frac{H}{h}}.$$

Similar strategy can be applied.
First we calculate in the usual way that

$$K_H^t U K_H^{-t} = q^{\frac{\sin t \cos t}{2}} V^{\sin t} U^{\cos t}$$

$$K_H^t V K_H^{-t} = q^{-\frac{\sin t \cos t}{2}} V^{\cos t} U^{-\sin t}$$

Pick up t such that

$$e = \sin t, \quad f = \cos t \quad g = ef^{-1}, \qquad e, f, g \in \mathbb{Q}$$

We work in a (U^f, V^g)-system, that is the identity

$$U^f V^g = q^e V^g U^f \quad (\text{use } e = fg)$$

Time evolution for Harmonic oscillator

$$K^t = e^{it\frac{H}{h}}.$$

Similar strategy can be applied.
First we calculate in the usual way that

$$K_H^t U K_H^{-t} = q^{\frac{\sin t \cos t}{2}} V^{\sin t} U^{\cos t}$$

$$K_H^t V K_H^{-t} = q^{-\frac{\sin t \cos t}{2}} V^{\cos t} U^{-\sin t}$$

Pick up t such that

$$e = \sin t, \quad f = \cos t \quad g = ef^{-1}, \qquad e, f, g \in \mathbb{Q}$$

We work in a (U^f, V^g)-system, that is the identity

$$U^f V^g = q^e V^g U^f \quad (\text{use } e = fg)$$

Direct calculation as above yield

$$\langle x_1|K^t|x_2\rangle =$$

$$= c_0\sqrt{\frac{h\cdot|\sin t|}{2\pi}}\exp i\frac{(x_1^2+x_2^2)\cos t - 2x_1x_2}{2h\sin t}$$

$$|c_0| = 1.$$

Or, in a different normalisation (if we replace sums by integrals)

$$c_0\frac{1}{\sqrt{2\pi h\cdot|\sin t|}}\exp i\frac{(x_1^2+x_2^2)\cos t - 2x_1x_2}{2h\sin t}$$

References

1. B.Zilber, The noncommutative torus and Dirac calculus (draft). http://people.maths.ox.ac.uk/ zilber/dirac.dvi

2. B.Zilber, Notes on Approximation (draft) http://people.maths.ox.ac.uk/ zilber/approx.dvi

3. B.Zilber, *A class of quantum Zariski geometries.* In: Model Theory with applications to algebra and analysis, I and II (Z. Chatzidakis, H.D. Macpherson, A. Pillay, A.J. Wilkie editors), Cambridge University Press, Cambridge.

4. B.Zilber, *Non-commutative Zariski geometries and their classical limit.* arXiv0707.0780[math.QA] June 2007