Stabilization of Concentration Profiles in Catalyst Particles

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INTRODUCTION

In a recent paper [1], Aronson and Peletier studied the global stability of concentration profiles in a one-dimensional model of a catalyst particle. They considered an infinite slab of homogeneous material with catalytic material situated on both of its faces. The slab is immersed in a bath in which the concentration of the reactant is kept at a constant value. This led to the study of the following mixed initial-boundary value problem

\[ u_t = u_{xx}, \quad \text{for } 0 < x < 1, \ t > 0, \]  
\[ u_x(i, t) = (-1)^i \lambda f(u(i, t)), \quad \text{for } i = 0, 1, \ t > 0, \]  
\[ u(x, 0) = \varphi(x), \quad \text{for } 0 \leq x \leq 1. \]

Here \( x \) denotes the spatial coordinate perpendicular to the faces of the slab, these being situated at \( x = 0 \) and \( x = 1 \). The variable \( t \) denotes time, \( u \) denotes a dimensionless concentration and \( \lambda \) denotes a positive parameter. The function \( f \) appearing in the boundary conditions is given by

\[ f(u) = [k_1u/(1 + k_2u)]^2 + u - 1, \]

in which \( k_1 \) and \( k_2 \) are suitably chosen positive constants. This function is related to the rate of consumption of the reactant by the catalytic material situated on the faces.

It was shown in [1] that for each \( \varphi \in C([0, 1]) \) problem (1)–(3) has a unique solution \( u(x, t; \varphi) \). It was also shown that, depending on the value of \( \lambda \), problem (1), (2) could have 3, 5, 7, or 9 equilibrium solutions. The main emphasis in [1] was on a discussion of the question of stability of these
equilibrium solutions. Specifically, if \( \bar{u}(x) \) is such an equilibrium solution, a partial characterization was given of the region of attraction \( A(\bar{u}) \), where

\[
A(\bar{u}) \overset{\text{def}}{=} \{ \psi \in C([0, 1]) : u(x, t, \psi) \to \bar{u}(x) \text{ uniformly on } [0, 1] \text{ as } t \to \infty \}.
\]

In this paper we shall be interested in the following question. Given any \( \psi \in C([0, 1]) \), must \( u(x, t; \psi) \) converge as \( t \to \infty \) to some equilibrium solution? That is, is it true that

\[
\bigcup_{j=1}^{n} A(\bar{u}_j) = C([0, 1]),
\]

where \( \{\bar{u}_j\}, j = 1, 2, \ldots, n \), is the set of equilibrium solutions? We shall show that this is indeed the case, and that in addition convergence to the relevant equilibrium solution holds in \( C^1([0, 1]) \).

In [1] the value of \( \lambda \) and the form of the function \( f \) played an important role in the characterization of the equilibrium solutions and their regions of attraction. In the present paper we shall not be interested in a detailed description of the equilibrium solutions. It will therefore be possible to obtain without extra effort the above result for the more general problem

\[
\begin{align*}
  u_t &= u_{xx}, & 0 < x < 1, \; t > 0, \\
  u_x(i, t) &= (-1)^i f_i(u(i, t)), & i = 0, 1, \; t > 0, \\
  u(x, 0) &= \psi(x), & 0 \leq x \leq 1,
\end{align*}
\]

in which \( f_0 \) and \( f_1 \) are twice continuously differentiable functions defined on \( \mathbb{R} \), each satisfying the following hypotheses:

(\text{H1}) There exists a positive constant \( a < \infty \) such that

\[
  s f(s) > 0 \quad \text{for } \quad |s| > a.
\]

As we shall see, this condition ensures that

(i) problem (4), (5) has at least one equilibrium solution;

(ii) \( u(x, t; \psi) \) is uniformly bounded for \( 0 \leq x \leq 1, \; t \geq 0 \).

(\text{H2}) The equilibrium solutions of problem (4), (5) are isolated in \( C([0, 1]) \).

To prove our result we will use the invariance principle discovered for ordinary differential equations by LaSalle and extended to general semigroups by Hale [9]. The earliest example of the use of similar techniques to study the asymptotic behavior of a partial differential equation seems to be the work of Zelenyak [11]. Invariance techniques have now been successfully applied to a number of problems involving partial differential equations (see, for example, [2, 8]). In particular, a result similar to ours for solutions of a one-
dimensional semilinear parabolic equation with zero Dirichlet data at the lateral boundary has been established by Rudenko [10], who used a result due to Zelenyak [12], and by Chafee and Infante [5]. Recently the same problem with zero Neumann data was treated by Chafee [4] who has also studied in [3] a related problem on an infinite interval.

The major burden of our work is to show that the solution \( u(x, t; \psi) \) has sufficient regularity properties for the invariance principle to be applied. In particular, we show that if \( \psi \in C([0, 1]) \) then \( u_t(., .; \psi) \in C([0, 1] \times [a, b]) \) for \( 0 < a < b < \infty \). This is done in Section I by considering the equivalent system of Volterra integral equations and by use of the maximum principle for the heat equation. Then in Section II we fairly rapidly prove the main result.

I

We first introduce some notation. We shall write

\[
Q_T = \{(x, t) : 0 < x < 1, 0 < t \leq T\},
\]

\[
S_T = \{(x, t) : x \in \{0, 1\}, 0 < t \leq T\},
\]

where \( T > 0 \) may be infinite. Let \( Q = Q_\infty \), \( S = S_\infty \). Denote by \( \bar{Q}_T \) the closure of \( Q_T \).

A function \( u = u(x, t; \psi) \) is said to be a solution of problem (4)-(6) if \( u \in C(Q), u_t \in C(Q \cup S), u_{xx} \in C(Q), u_t \in C(Q) \), and (4)-(6) hold. For problem (1)-(3), which can be reduced to a special case of (4)-(6), existence and uniqueness was established in [1]. However, examination of the proof reveals that the only properties of the function \( f \) in (2) which were needed were that \( f \in C^\alpha(\mathbb{R}) \) and a property which ensured that \( u(x, t; \psi) \) is uniformly bounded in \( \bar{Q} \). In problem (4)-(6) it is hypothesis (H1) which takes care of the boundedness of \( u \). To prove this we use a slight variation of a maximum principle established in [1].

**Lemma 1.** Let \( v_j(x) = p_j + (q_j - p_j)x \) for \( j = 1, 2 \). Assume that

\[
q_1 \geq p_1 + f_0(p_1), \\
p_1 \geq q_1 + f_1(q_1),
\]

and

\[
q_2 \leq p_2 + f_0(p_2), \\
p_2 \leq q_2 + f_1(q_2).
\]

If \( \psi \) satisfies

\[
v_1 \leq \psi \leq v_2 \quad \text{on} \quad [0, 1],
\]

then

\[
v_1(x) \leq u(x, t; \psi) \leq v_2(x) \quad \text{in} \quad \bar{Q}.
\]

For the proof we refer to [1].
Given any $\psi \in C([0, 1])$ there exists a constant $u^*$ satisfying

(i) $u^* \geq \max\{a_0, a_1\}$, where $a_i$ is the constant defined in hypothesis (H1) for the function $f_i$ ($i = 0, 1$), and

(ii) $u^* \geq |\psi(x)|$ for all $x \in [0, 1]$.

By Lemma 1 and hypothesis (H1),

$$-u^* \leq u(x, t; \psi) \leq u^* \quad \text{in } Q \text{ for all } t \geq 0.$$ (7)

By (7) the asymptotic behavior of $u(x, t; \psi)$ is unaffected by the values of $f_i$ outside $[-u^*, u^*]$.

Thus, since $f_i \in C^2(\mathbb{R})$, without loss of generality we may and shall assume that there are constants $M_j$ ($j = 0, 1, 2$) such that for $i = 0$ and $1$,

$$|f_i(s)| \leq M_0, \quad |f'_i(s)| \leq M_1, \quad |f''_i(s)| \leq M_2 \quad \text{for all } s \in \mathbb{R},$$

where primes denote differentiation.

One can now proceed as in [1] to prove the following result.

**Theorem 1.** Let $\psi \in C([0, 1])$. Then problem (4)--(6) possesses a unique solution. Moreover, for each $T > 0$ there exists a constants $C_T$ such that

$$\max_{\partial_T} |u(x, t; \psi_1) - u(x, t; \psi_2)| \leq C_T \max_{[0, 1]} |\psi_1(x) - \psi_2(x)|$$

for any $\psi_1, \psi_2 \in C([0, 1])$.

Let $u(x, t; \psi)$ be the solution of problem (4)--(6). By means of the following lemma we can reduce our problem to one only involving the two functions $u(0, t)$ and $u(1, t)$. (For convenience we shall sometimes omit reference to $\psi$.)

**Lemma 2.** Let $u(0, t)$ and $u(1, t)$ belong to $C^4(0, \infty)$. Then $u_i \in C([0, 1] \times [a, b])$ for $0 < a < b < \infty$.

**Proof.** Let $G(x, \xi, t)$ be the Green function for the heat equation on $(0, 1) \times (0, \infty)$ with zero Neumann data. It can be given explicitly by

$$G(x, \xi, t) = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\pi^2t} \cos nx \cos n\pi \xi.$$

Then the solution $u$ of problem (4)--(6) can be written as

$$u(x, t) = \int_0^t G(x, \xi, t) \psi(\xi) d\xi - \frac{1}{2} \sum_{i=0}^{\infty} \int_0^t G(x, i, t-\tau) f_i(u(i, \tau)) d\tau. \quad (8)$$
To begin with, we assume that $u(i, t) \in C^1([0, \infty))$ for $i = 0, 1$. We substitute $\tau = t - s$ in (8) and differentiate with respect to $t$. This yields

$$u_i(x, t) = \int_0^1 G_i(x, \xi, t) \psi(\xi) \, d\xi - \sum_{i=0}^1 \int_0^t G(x, i, t) f_i(u(i, 0))$$

$$- \sum_{i=0}^1 \int_0^t G(x, i, s) f_i'(u(i, t - s)) u'(i, t - s) \, ds.$$  

If $0 < a < b < \infty$ we can use the expression for $G$ to write the first term as a uniformly convergent series of functions in $C([0, 1] \times [a, b])$. Hence this term belongs to $C([0, 1] \times [a, b])$.

The second term clearly belongs to $C([0, 1] \times [a, b])$. To treat the third term, note that both $f_i'(u(i, t - s))$ and $u'(i, t - s)$ are bounded for $0 \leq s \leq t$. Moreover, for $x, \xi \in [0, 1]$,

$$\int_0^t |G(x, \xi, t)| \, ds \leq t + 2 \sum_{n=1}^{\infty} \left((1 - e^{-n^2\pi^2})/n^2\pi^2\right).$$

Hence this term can also be expressed as a uniformly convergent series of functions belonging to $C([0, 1] \times [a, b])$ and therefore belongs itself to $C([0, 1] \times [a, b])$.

Thus we have shown that $u_i \in C([0, 1] \times [a, b])$ if $u'(i, t)$ is continuous up to $t = 0$ for $i = 0$ and 1. It remains to dispose of this last condition. Instead of problem (4)-(6) we consider the problem (4), (5) with initial value

$$u_i(t) = u(x, \tau; \psi)$$

for some $\tau > 0$. Since $u(x, t; \psi) = u(x, t + \tau; \psi)$ it follows that $u(i, t; \tilde{\psi}) \in C^1([0, \infty))$. By the first part of the proof $u_i(\cdot; \psi) \in C([0, 1] \times [a, b])$ for $0 < a < b < \infty$ and hence $u_i(\cdot; \psi) \in C([0, 1] \times [a, b])$ for $\tau < a < b < \infty$.

Since we may choose $\tau$ as small as we wish, it follows that $u_i(\cdot; \psi) \in C([0, 1] \times [a, b])$ whenever $0 < a < b < \infty$.

It follows from (8) that the functions $u_i(t) = u(i, t; \psi)$ satisfy the pair of Volterra integral equations

$$u_i(t) = \int_0^1 G(i, \xi, t) \psi(\xi) \, d\xi - \sum_{i=0}^1 \int_0^t g_i(t - \tau) f_i(u_i(\tau)) \, d\tau, \quad i = 0, 1,$$

where $g_i(t) = G(i, j, t)$. To write these equations more compactly, we introduce the vector-valued functions $u(t) = (u_0(t), u_1(t))$, $f(u) = (f_0(u_0), f_1(u_1))$ and $\phi(t) = (\phi_0(t), \phi_1(t))$, where $\phi_i(t) = \int_0^1 G(i, \xi, t) \psi(\xi) \, d\xi$, and the matrix $G(t) = (g_{ij}(t))$. We then obtain

$$u(t) = \phi(t) - \int_0^t G(t - \tau) f(u(\tau)) \, d\tau. \quad (9)$$
Let $I$ be an interval on the real line, and let $\mathcal{C}^k(I)$ be the set of functions $I \to \mathbb{R}^3$ which, together with their first $k$ derivatives, are continuous on $I$. In [1] it was shown that (9) has a unique solution $u$, which for every $T > 0$ belongs to $\mathcal{C}^3([0, T])$. It follows from Lemma 2 that it will be enough to prove that in addition $u$ belongs to $\mathcal{C}^3([0, T])$. It will be helpful to prove the following lemma, which is also of critical importance for the analysis in Section II.

**Lemma 3.** Let $\psi \in C([0, 1])$ and let $\delta > 0$. There exists a constant $k > 0$ such that $|u_x(x, t; \psi)| \leq k$ for all $x \in [0, 1]$ and $t \geq \delta$.

**Proof.** We have already shown that $u(x, t; \psi)$ is uniformly bounded on $\bar{Q}$, and thus on $S$. In view of the boundary conditions (5) this implies that $u_x$ is bounded on $S$. Moreover, since $\delta > 0$, $u(\cdot, \delta; \psi) \in C^3([0, 1])$. Thus $u_x$ is bounded on the parabolic boundary of the cylinder $[0, 1] \times [\delta, \infty)$. Since $u_x$ satisfies the heat equation in $Q$, it follows by the maximum principle that $u_x$ is bounded in $[0, 1] \times [\delta, \infty)$ as required.

Next we estimate the behavior of $\phi(t)$ and $\phi'(t)$. Because $G(x, \xi, t)$ is singular at $t = 0$, we may also expect $\phi'(t)$ to be singular at $t = 0$. Let

$$f(x, t) = [1/(2\pi t)^{1/2}] e^{-x^2/4t}, \quad x \in \mathbb{R}, \ t > 0,$$

denote the source function for the heat equation in one dimension.

Let $\psi \in C([0, 1])$ and let $\tilde{\psi}$ denote the unique extension of $\psi$ to $\mathbb{R}$ which is symmetric with respect to $x = 0$ and $x = 1$. Then

$$\int_0^1 G(x, \xi, t) \psi(\xi) d\xi = \int_{-\infty}^\infty f(x - \xi, t) \tilde{\psi}(\xi) d\xi, \quad x \in [0, 1], t > 0. \quad (10)$$

It follows that $\phi \in C([0, \infty))$, and that

$$|\phi_i(t)| \leq \max_{[0, 1]} |\psi(x)|, \quad t \geq 0, \quad i = 0, 1. \quad (11)$$

Also, if $\psi \in C^3([0, 1])$,

$$\phi_i'(t) = -\int_{-\infty}^{\infty} f(t - \xi, t) \tilde{\psi}(\xi) d\xi,$$

and thus, by a routine computation,

$$|\phi_i'(t)| \leq [1/(\pi t)^{1/2}] \max_{[0, 1]} |\psi'(x)|, \quad t > 0, \quad i = 0, 1. \quad (12)$$

We also need an estimate for the behavior of the functions $g_{\psi}(t)$ as $t \to 0$.

From (10) we see that

$$G(x, \xi, t) = \sum_{n=-\infty}^{\infty} \{f(x - \xi - 2n, t) + f(x + \xi - 2n, t)\}.$$
Hence
\[ |g_{i}(t)| \leq \omega(t), \quad t > 0, \]
where
\[ \omega(t) \equiv G(0, 0, t) = 2 \sum_{n=-\infty}^{\infty} j(-2n, t). \]

An elementary computation now shows that (i) \( \omega(t) \) is continuous and non-increasing for \( t > 0 \), and (ii) \( \omega(t) \sim (\pi t)^{-1/2} \) as \( t \to 0^+ \).

For future reference we introduce three functions \( h_i(t) \) \( i = 1, 2, 3 \) which are related to \( \omega(t) \). Let \( \alpha \in (\frac{1}{2}, 1) \) and
\[
\begin{align*}
    h_1(t) &= \int_{0}^{t} \omega(s) \, ds, \quad \text{for} \quad t > 0, \\
    h_2(t) &= \sup_{(0,t)} s^{\alpha} \omega(s), \quad \text{for} \quad t > 0, \\
    h_3(t) &= \sup_{(0,t)} \int_{0}^{t} \omega(s - r) r^{-\alpha} \, dr, \quad \text{for} \quad t > 0.
\end{align*}
\]

It is clear from properties (i) and (ii) of \( \omega \) that the functions \( h_i \) are well defined. Moreover, \( h_i \to 0 \) for \( t \to 0^+ \) and \( i = 1, 2, 3 \). The value of \( \alpha \) will be fixed throughout our discussion.

In view of the singular behavior of \( \phi' \) we shall discuss (9) in a weighted space of continuous functions.

**DEFINITION.** Let \( \gamma > 0 \). We denote by \( X(\gamma) \) the space of functions \( \xi \in C([0, \gamma]) \cap C^1((0, \gamma)) \) such that
\[ \| \xi \|_{X} = \sum_{i=0}^{1} \{ \sup_{(a,\gamma)} | \xi_i(t) | + \sup_{(0,\gamma)} | t^\alpha \xi'_i(t) | \} < \infty. \]

It is not difficult to show that \( (X(\gamma), \| \cdot \|_{X}) \) is a Banach space. We shall frequently omit reference to \( \gamma \).

**THEOREM 2.** Let \( \psi \in C([0, I]) \). Then \( u(t) = (u_0(t), u_1(t)) \) belongs to \( C^1([0, T]) \) for every \( T > 0 \).

**Proof.** By an argument similar to that at the end of the proof of Lemma 2 it is clear that without loss of generality we may suppose that \( \psi \in C^1([0, I]) \).

Now define the operator \( A \) by
\[
(Au)(t) \equiv \int_{0}^{t} G(t - \tau) f(u(\tau)) \, d\tau.
\]
Then (9) can be written as

\[ u = \phi - Au. \]

We shall first show that the operator

\[ Ku = \phi - Au \]

is a contraction on a certain closed ball of \( X(\gamma) \) for \( \gamma \) sufficiently small. The existence and uniqueness of a solution of (9) in \( X(\gamma) \) then follows from the well-known Banach fixed point theorem. We will then show how this argument may be repeated to show that \( u(t) \in C^3(0, T) \) for any \( T > 0 \).

Since \( \alpha > \frac{1}{2} \), it follows from (12) that \( \phi \in X \). We now show that because \( \alpha < 1 \) the operator \( A \) is defined and bounded on \( X \). Let \( u \in X \). Then we have for \( i = 0, 1 \) and \( t \in (0, \gamma) \)

\[ |(Au)_i(t)| = \left| \sum_{j=0}^{1} \int_0^t g_s(t - \tau) f_j(u_j(\tau)) \, d\tau \right| \leq 2M_0h_1(\gamma), \]

and hence

\[ \| Au \|_\infty \leq 4M_0h_1(\gamma), \]

where

\[ \xi = \sum_{i=1}^{1} \sup_{(x,v)} | \xi_i(t)|. \]

Moreover, it follows after a straightforward computation that for \( i = 0, 1 \):

\[ (Au)_i(t) = \sum_{j=0}^{1} g_s(t) f_j(u_j(0)) + \sum_{j=0}^{1} \int_0^t g_s(t - \tau) f_i'(u_i(\tau)) u_j(\tau) \, d\tau. \]

Hence

\[ | t^a(Au)_i(t) | \leq 2M_0t^a \omega(t) + M_1t^a \int_0^t \omega(t - \tau) \tau^{-\alpha} \, d\tau \cdot \| t^a u' \|_\infty \]

and therefore

\[ \| t^a(Au)' \|_\infty \leq 4M_0h_2(\gamma) + 2M_1h_3(\gamma) \| t^a u' \|_\infty. \]

Thus

\[ \| Au \|_X \leq 4M_0(h_1 + h_2) + 2M_1h_3(\| \phi \|_X + R). \quad (13) \]

Suppose that \( R > 0 \) and \( \| u - \phi \|_X \leq R \). Then it follows from (13) that

\[ \| Ku - \phi \|_X \leq 4M_0(h_1 + h_2) + 2M_1h_3(\| \phi \|_X + R). \]

Hence, because \( h_i(\gamma) \to 0 \) and \( \| \phi \|_X \) does not increase as \( \gamma \to 0^+ \), there
exists $\gamma_0 > 0$ such that $\| Ku - \phi \|_X \leq R$ if $\gamma \leq \gamma_0$. Thus, if $\gamma \leq \gamma_0$, $K$ maps the closed ball

$$\bar{B}_R(\phi) = \{ \zeta \in X : \| \zeta - \phi \|_X \leq R \}$$

into itself.

We next show that $K$ is a contraction for sufficiently small values of $\gamma$. This will be so if $A$ is a contraction for small $\gamma$.

Let $u, v \in \bar{B}_R(\phi)$. Then we obtain, using the mean value theorem and the bound for $g_{ij}$:

$$\| (Au)_i(t) - (Av)_i(t) \|_{\bar{\Phi}} \leq M_1 \sum \int_0^t \omega(t - \tau) \| u_j(\tau) - v_j(\tau) \| d\tau$$

$$\leq M_1 h_1(\gamma) \| u - v \|_{\bar{\Phi}}, \quad i = 0, 1,$$

when $0 < t < \gamma$. Hence

$$\| Au - Av \|_{\bar{\Phi}} \leq 2M_1 h_1(\gamma) \| u - v \|_{\bar{\Phi}}.$$

Similarly, we obtain

$$\| t^\omega (Au)' - t^\omega (Av)' \|_{\bar{\Phi}} \leq 2(M_1 h_2 + RM_1 h_3 + \| \phi \|_X M_2 h_3) \| u - v \|_{\bar{\Phi}}$$

$$+ 2M_1 h_3 \| t^\omega u' - t^\omega v' \|_{\bar{\Phi}}.$$

Therefore

$$\| Au - Av \|_X \leq L(\gamma) \| u - v \|_X,$$

where

$$L(\gamma) = \max \{ 2M_1 h_1(\gamma) + h_3(\gamma) \} + 2M_2 [R + \| \phi \|_X h_3(\gamma), 2M_4 h_3(\gamma) \}.$$

Because $h_1(\gamma) \to 0$ and $\| \phi \|_X$ does not increase as $\gamma \to 0+$, $L(\gamma) \to 0$ as $\gamma \to 0+$ and there exists a number $\gamma_1 > 0$ such that $L(\gamma) < 1$ if $\gamma \leq \gamma_1$. Thus if $\gamma \leq \gamma^* = \min\{\gamma_0, \gamma_1\}$, the operator $K$ is a contraction which maps $\bar{B}_R(\phi)$ into itself.

We now note that the above argument establishes that $\gamma^* < 1$ may be chosen so that, for any $\phi$ in a bounded set of $X(1)$, $K$ maps $\bar{B}_R(\phi)$ into itself and is a contraction. We also note that if we replace $\psi$ in (9) by $\bar{\psi} = u(\cdot, \tau; \psi)$ for any $\tau \geq 0$, then by Lemma 3 and the estimates (11), (12), the corresponding functions $\bar{\phi}$ are bounded in $X(1)$ independently of $\tau \geq 0$. Hence the above argument establishes that $u(t) \in \mathcal{C}(\tau, \tau + \gamma^*)$ for any $\tau \geq 0$, and the desired result follows.

**Corollary.** Let $\psi \in C([0, 1])$. Then $u(t, \cdot; \psi) \in C([0, 1] \times [a, b])$ for $0 < a < b < \infty$.

**Proof.** This is immediate from Lemma 2.
Let \( u(x, t; \psi) \) be the solution of the problem (4)–(6),

\[
\begin{align*}
  u_t &= u_{xx}, & (x, t) &\in Q, \\
  u_x(i, t) &= (-1)^i f_i(u(i, t)), & i &= 0, 1, & t &> 0, \\
  u(x, 0) &= \psi(x), & 0 &\leq x &\leq 1,
\end{align*}
\]

in which the functions \( f_i \) satisfy conditions (H1) and (H2), and \( \psi \in C([0, 1]) \).

Define the operators \( T(t) : C([0, 1]) \to C([0, 1]) \) by

\[
T(t)\psi = u(\cdot; t; \psi), \quad t \geq 0.
\]

It follows from Theorem 1 that \( \{T(t)\} \ t \geq 0 \) is a semigroup on \( C([0, 1]) \); that is, (i) \( T(0) = \text{identity} \), and (ii) \( T(s)T(t) = T(s+t) \) for all \( s, t \geq 0 \).

**Lemma 4.** Let \( \psi \in C([0, 1]) \) and \( \tau > 0 \). Then the set \( \{T(t)\psi : t \geq \tau\} \) is precompact in \( C([0, 1]) \).

**Proof.** It was shown in Section I that there is a constant \( K_0 \) such that

\[
\|T(t)\psi\|_0 \leq K_0, \quad \text{for} \quad t \geq 0, \tag{14}
\]

where \( \| \cdot \|_0 \) denotes the supremum norm in \( C([0, 1]) \). Moreover, by Lemma 3, \( u_x(x, t; \psi) \) is uniformly bounded for \( x \in [0, 1] \) and \( t \geq \tau \). Hence the set \( \{T(t)\psi : t \geq \tau\} \) is bounded and equicontinuous, and thus precompact by Ascoli’s theorem.

In the usual way we define the norm of an element \( \zeta \in C^1([0, 1]) \) by

\[
\|\zeta\|_1 = \|\zeta\|_0 + \|\zeta'\|_0.
\]

We need the following continuity properties of the semigroup \( \{T(t)\} \ t \geq 0 \).

**Lemma 5.** (a) For \( t > 0 \), \( T(t) \) is a continuous map from \( C([0, 1]) \) into \( C^1([0, 1]) \). (b) For each \( \psi \in C([0, 1]) \) the map \( T(\cdot)\psi : (0, \infty) \to C^1([0, 1]) \) is continuous.

**Proof.** First note that (b) follows immediately from (a) and the known continuity of \( T(\cdot)\psi : (0, \infty) \to C([0, 1]) \).

To prove (a) let \( \psi_n \to \psi \) in \( C([0, 1]) \). Let \( t > 0 \), and set \( u(t) = T(t)\psi \), \( u_n(t) = T(t)\psi_n \). Then by Theorem 1,

\[
\|u_n(t) - u(t)\|_0 \to 0 \quad \text{as} \quad n \to \infty
\]

and we need to show that

\[
\|u_{nx}(\cdot, t) - u_x(\cdot, t)\|_0 \to 0 \quad \text{as} \quad n \to \infty.
\]
It follows from (8) that
\[
  u_{nx}(x, t) - u_x(x, t) = \int_0^1 G_x(x, \xi, t)[\psi_n(\xi) - \psi(\xi)] d\xi
  - \sum_{i=0}^1 \int_0^t G_x(x, i, t - \tau)[f_i(u_n(i, \tau)) - f_i(u(i, \tau))] d\tau.
\]

Let \( \tilde{\psi}_n \) and \( \tilde{\psi} \) be the periodic extensions of, respectively, \( \psi_n \) and \( \psi \) into \( \mathbb{R} \) such that \( \tilde{\psi}_n \) and \( \tilde{\psi} \) are symmetric with respect to \( x = 0 \) and \( x = 1 \). Then the first term in the expression for \( u_{nx} - u_x \) can be written as
\[
  I_1 = \int_{-\infty}^{\infty} J_x(x - \xi, t)(\tilde{\psi}_n(\xi) - \tilde{\psi}(\xi)) d\xi.
\]

Hence
\[
  |I_1| \leq \int_{-\infty}^{\infty} |J_x(x - \xi, t)| d\xi \cdot \sup_{\mathbb{R}} |\tilde{\psi}_n(\xi) - \tilde{\psi}(\xi)| = (\pi t)^{-1/2} \| \psi_n - \psi \|_0 .
\]

An elementary computation shows that the two integrals in (16) are bounded for \( 0 < t < T \). Hence there exists a constant \( K \) such that
\[
  |I_2| \leq M_1 \sum_{i=0}^1 \int_0^t |G_x(x, i, t - \tau)| d\tau \cdot \max_{\mathcal{O}_T} |u_n(x, t) - u(x, t)|
  \leq M_1 \sum_{i=0}^1 \int_0^t |G_x(x, i, t - \tau)| d\tau \cdot C_T \| \psi_n - \psi \|_0 .
\]

Define the function \( V: C^1([0, 1]) \to \mathbb{R} \) by
\[
  V(\xi) = \frac{1}{2} \int_0^1 (\xi'(x))^2 dx + \sum_{i=0}^1 F_i(\xi(i)),
\]
where
\[
  F_i(\xi) = \int_0^{\xi_i} f_i(q) dq, \quad i = 0, 1.
\]

It is readily shown that \( V \) is continuous.
Lemma 6. Let \( \psi \in C([0, 1]) \) and let \( 0 < \tau < t < \infty \). Then

\[
V(T(t)\psi) - V(T(\tau)\psi) = - \int_\tau^t ds \int_0^1 u_t^2(x, s; \psi) \, dx. \tag{17}
\]

Proof. Let \( 0 < \delta < \frac{1}{2} \). Following Chafee [4], define \( V_\delta : C^1([0, 1]) \to \mathbb{R} \) by

\[
V_\delta(\xi) = \frac{1}{2} \int_\delta^{1-\delta} (\xi'(x))^2 \, dx + F_0(\xi(0)) + F_1(\xi(1)).
\]

Since \( u(x, t; \psi) = T(t)\psi \) satisfies the heat equation, it is smooth in \( Q \). Therefore for \( t > 0 \),

\[
\left( \frac{d}{dt} \right) V_\delta(T(t)\psi) = u_{t+} |_{x=\delta}^{x=1-\delta} - \int_\delta^{1-\delta} u_t^2 \, dx + \left( \frac{d}{dt} \right) [F_0(u(0, t; \psi)) + F_1(u(1, t; \psi))].
\]

Hence if \( 0 < \tau < t < \infty \),

\[
V_\delta(T(t)\psi) - V_\delta(T(\tau)\psi) = \int_\tau^t u_{t+} |_{x=\delta}^{x=1-\delta} ds - \int_\delta^{1-\delta} u_t^2 \, dx
\]

\[
+ [F_0(u(0, s; \psi)) + F_1(u(1, s; \psi))]_{x=\tau}^{x=t}.
\]

Let \( \delta \to 0^+ \). Then by Lemma 2 and the dominated convergence theorem we obtain (17).

For \( \psi \in C([0, 1]) \) define the \( \omega \)-limit set \( \omega(\psi) \) by \( \omega(\psi) = \{ \chi \in C^1([0, 1]) : \text{there exists } \{t_n\}, \quad t_n \to \infty \text{ as } n \to \infty, \text{ with } T(t_n)\psi \to \chi \text{ in } C^1([0, 1]) \} \). We can now prove our main result:

**Theorem 3.** Let \( \psi \in C([0, 1]) \). Then, as \( t \to \infty \), \( T(t)\psi \to \nu \) in \( C^1([0, 1]) \), where \( \nu \) is an equilibrium solution.

Proof. Without loss of generality we may assume that \( \psi \in C^1([0, 1]) \). By Lemma 5 \( \{T(t)\}_{t \geq 0} \) defines by restriction a semigroup of continuous operators on \( C^1([0, 1]) \) such that for every \( \psi \in C^1([0, 1]) \) the map \( T(\cdot)\psi : (0, \infty) \to C^1([0, 1]) \) is continuous.

Since \( u(x, t; \psi) \) is bounded, \( V(T(t)\psi) \) is bounded below for \( t \geq 0 \). Also, by Lemma 6, \( V(T(t)\psi) \) is nonincreasing for \( t > 0 \). Bearing in mind Lemma 4 it follows from [9] that \( \omega(\psi) \) is nonempty, positively invariant (i.e., \( T(t) \omega(\psi) \subseteq \omega(\psi) \) for \( t \geq 0 \)) and connected. Furthermore, as \( t \to \infty \), \( d(T(t)\psi, M) \to 0 \), where \( d \) denotes distance in \( C^1([0, 1]) \) and where \( M \) is the largest positively invariant set contained in \( \{ \chi \in C^1([0, 1]) : V(\chi) = \inf_{t \geq 0} V(T(t)\psi) \} \).
By Lemma 6, $M$ contains only equilibrium solutions. Since these solutions are by hypothesis (H2) isolated, it follows that $\omega(\psi) = \{v\}$ for some equilibrium solution $v$, and that $T(t)\psi \to v$ in $C^1([0, 1])$ as $t \to \infty$.

**Remark.** In [9] it was assumed (partly so as to obtain stronger conclusions than we require) that the map $(t, \psi) \to T(t)\psi$ is jointly continuous on $(0, \infty) \times C^1([0, 1])$, whereas we have established only separate continuity with respect to $t$ and $\psi$. This apparent restriction was removed by Dafermos [7]; Chernoff and Marsden [6] have shown, however, that for a semigroup defined on a metric space joint continuity is implied by separate continuity. For our problem joint continuity is easy to prove directly.

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**References**