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Universal singular sets for one-dimensional variational problems

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Summary. A study is made of the regularity properties of minimizers u of the integral $I(u) = \int_a^b f(x,u,u')\,dx$ subject to the boundary conditions $u(a) = \alpha,\ u(b) = \beta$ as the interval (a,b) and boundary values α, β are varied. Under natural hypotheses on f it is shown that the set of points in the $(x,u)-$plane at which a minimizer u can have infinite derivative for some interval and boundary values is small in the sense of category.

1 Introduction

We consider the problem of minimizing the integral

$$I(u) = \int_a^b f(x, u(x), u'(x))\ \mathrm{d}x \tag{1.1}$$

among absolutely continuous real-valued functions u on [a,b] with given boundary values $u(a) = \alpha,\ u(b) = \beta$.

We assume throughout that $f = f(x,u,p)$ is a nonnegative C^3 function on $\mathbb{R}^3$ satisfying $f_{pp} > 0$ and the superlinear growth condition

$$f(x,u,p) \geq \phi(p)\ , \tag{1.2}$$

where $\varphi(p)/|p| \to \infty$ as $|p| \to \infty$. According to classical results of Tonelli [7, II pp. 282, 359], [8] under these assumptions the minimum of I subject to the given boundary conditions is attained, and any minimizer u has a finite or infinite derivative u' at every point of [a,b]. Moreover $u' : [a,b] \to \mathbb{R} \cup \{-\infty, \infty\}$ is continuous and $E := \{x \in [a,b] : |u'(x)| = \infty\}$ is a closed set of measure zero. (For a review of these results see Ball and Mizel [1].)

In general E is non-empty; examples with f a polynomial can be found in Ball and Mizel [1]. For some recent results involving similar examples see Sychev [5, 6]. Furthermore, Tonelli's partial regularity theorem described above is optimal in the sense that any closed set of measure zero is the singular set E corresponding to a suitable f and boundary conditions (Davie [3]).

Since u is absolutely continuous it follows that meas $u(E) = 0$, and hence that the subset

$$D = \{(x, u(x)) : x \in E\}$$

of $\mathbb{R}^2$ is closed and has two-dimensional Lebesgue measure zero. Of course for a given f the set D is in general dependent on the interval (a, b), the boundary values α, β and the minimizer u, and the question therefore arises as to what happens as these are varied. In this paper we show (Theorem 2.1) that the set of points $\mathscr{D}_f$ of $\mathbb{R}^2$ that belong to D for some a, b, α, β and minimizer u is of first category. If the interval is fixed and the boundary values are restricted by the inequalities $|\alpha| \leq M$, $|\beta| \leq M$ then (Theorem 2.2) the closure of the set of such points is nowhere dense. Thus most points of $\mathbb{R}^2$ cannot support a singularity for any interval or boundary conditions. We call $\mathscr{D}_f$ the *universal singular set for f.*

A special case of Theorem 2.1 can be deduced from the result of Clarke and Vinter [2] who show that if in addition to the hypotheses of Theorem 2.1 $f = f(x, u, p)$ is a polynomial in p of the form

$$f(x, u, p) = \sum_{i=0}^{N} q_i(x, u)p^{N-i} \tag{1.3}$$

then

$$\mathscr{D}_f \subset \{(x, u) : q_0(x, u) = 0\}.$$

Thus if, for example, q_0 is a non-zero polynomial in x, u then $\mathscr{D}_f$ is nowhere dense. They deduce this result from the form of the Euler-Lagrange equation using a criterion of Tonelli [7, II p. 364], a method that does not work under our more general hypotheses.

In Sect. 3 we discuss whether analogous universal singular sets could be small for multi-dimensional problems of the calculus of variations.

2 The universal singular set

We denote by $W^{1,1}(a, b)$ the usual Sobolev space of (equivalence classes of) functions $u : (a, b) \to \mathbb{R}$ such that $\|u\|_{1,1} := \int_a^b(|u| + |u'|)dx < \infty$. As is well known, $W^{1,1}(a, b)$ consists precisely of those u having a representative that is absolutely continuous on [a,b], and we always choose this representative.

Let [a,b] $\subset \mathbb{R}$. By a *minimizer of I on* [a,b] we mean a function $u \in W^{1,1}(a, b)$ which for some α, β minimizes

$$I_{(a,b)}(v) = \int_a^b f(x, v, v')dx \tag{2.1}$$

in the set $\mathscr{A}_{(a,b)} = \{v \in W^{1,1}(a, b) : v(a) = \alpha, v(b) = \beta\}$. We define the *universal singular set for f* as the set $\mathscr{D}_f$ of points $(a, \alpha) \in \mathbb{R}^2$ such that there exists $b \neq a$ and a minimizer u of I on [a,b] (or on [b,a] if $b < a$) with $u(a) = \alpha$, $|u'(a)| = \infty$. Our main result is that $\mathscr{D}_f$ is small in the sense of category.

Theorem 2.1. *The set $\mathscr{D}_f$ is of first category in $\mathbb{R}^2$.*

If we restrict attention to minimizers u defined on a fixed interval $J = [A, B]$, $B > A$, with a bound on the values of $u(A)$, $u(B)$ then more can be said. Given $M > 0$ define $\mathscr{D}_{f,J,M}$ as the set of points $(a, \alpha) \in J \times \mathbb{R}$ such that there exists a minimizer u of I on J with $u(a) = \alpha$, $|u'(a)| = \infty$, $|u(A)| \leq M$ and $|u(B)| \leq M$.

Theorem 2.2. *The set $\overline{\mathscr{D}_{f,E,M}}$ is nowhere dense.*

Lemma 2.1. *Let u_j, $j = 1, 2, \ldots$, be minimizers of I on $[a, b]$ and suppose $u_j \to u$ uniformly on $[a, b]$. Then u is a minimizer of I on $[c, d]$ whenever $a < c < d < b$.*

Remark. It is not true in general that under the hypotheses of the lemma u is a minimizer of I on [a,b]. For example, consider the integral

$$I(u) = \int_0^1 [(x^2 - u^3)^2 (u')^{16} + \varepsilon (u')^2] \, \mathrm{d}x \tag{2.2}$$

where $\varepsilon > 0$ is sufficiently small (cf. Ball and Mizel [1]). Let u_j minimize I in the set

$$\mathscr{A}_j = \left\{ v \in W^{1,1}(0,1) : v(0) = -\frac{1}{j}, v(1) = 1 \right\}.$$

The minimizer u_j exists by Tonelli's theorem. Moreover, setting $u(x) = -\frac{1}{j} + (1 + \frac{1}{j})x$ we see that

$$I(u_j) \le c_0 + c_1 \varepsilon \tag{2.3}$$

for all j, where c_0 and c_1 are positive constants. In particular the u_j are bounded in $W^{1,2}(0,1)$, and so a subsequence converges uniformly in [0,1] to a function $u \in W^{1,2}(0,1)$ with $u(0) = 0$, $u(1) = 1$. To show that u is not a minimizer of I on [0,1], fix k_1, k_2 with $0 < k_1 < k_2 < 1$. Let $v \in W^{1,1}(0,1)$, $v(1) = 1$, with $v(x_1) \le k_1 x_1^{2/3}$ for some $x_1 \in (0,1)$, and let $x_1(v)$ be the largest such value of x_1. Then $k_1 x^{2/3} \le v(x) \le k_2 x^{2/3}$ on some maximal interval $[x_1(v), x_2(v)] \subset (0,1)$ and so

$$\begin{aligned} I(v) &\ge \int_{x_1(v)}^{x_2(v)} (x^2 - v^3)^2 (v')^{16} \, \mathrm{d}x \\ &\ge (1 - k_2^3)^2 \int_{x_1(v)}^{x_2(v)} x^4 (v')^{16} \mathrm{d}x. \end{aligned} \tag{2.4}$$

Setting $z = x^{11/15}$ we deduce using Jensen's inequality that

$$\begin{aligned} I(v) &\ge \left(\frac{11}{15}\right)^{15} (1 - k_2^3)^2 \int_{x_1(v)^{11/15}}^{x_2(v)^{11/15}} \left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^{16} \mathrm{d}z \\ &\ge \left(\frac{11}{15}\right)^{15} (1 - k_2^3)^2 \frac{(k_2 x_2(v)^{2/3} - k_1 x_1(v)^{2/3})^{16}}{(x_2(v)^{11/15} - x_1(v)^{11/15})^{15}} \\ &\ge c_2 x_1(v)^{-1/3}, \end{aligned} \tag{2.5}$$

where $c_2 > 0$ depends only on k_1, k_2. Applying (2.5) to u_j we deduce from (2.3) that $x_1(u_j) \ge c_3 > 0$, where c_3 depends for all sufficiently small ε only on k_1, k_2. Therefore

$$x_1(u) \ge c_3 > 0. \tag{2.6}$$

Hence by (2.5) applied to u,

$$I(u) \ge c_2. \tag{2.7}$$

But if $\varepsilon < \frac{3}{4}c_2$ then $I(x^{2/3}) = \frac{4\varepsilon}{3} < I(u)$, so that u is not a minimizer of I on [0,1].

Proof of Lemma 2.1. Let $a < c < d < b$, and let $v \in W^{1,1}(c,d)$ with $v(c) = u(c)$, $v(d) = u(d)$. We must show that $I_{(c,d)}(v) \geq I_{(c,d)}(u)$.

Since $u_j(a)$, $u_j(b)$ are bounded and the u_j are minimizers, it follows that $I_{(a,b)}(u_j)$ is bounded, and thus so is $\int_a^b \varphi(u_j')\,dx$. By the de la Vallée Poussin criterion it follows that $u \in W^{1,1}(a,b)$ and $u_j \rightharpoonup u$ in $W^{1,1}(a,b)$. In particular u is differentiable a.e. in (a,b). Pick points $x_0 \in (a,c)$, $x_1 \in (d,b)$ at which u is differentiable. Thus there exists $M > 0$ such that

$$\left|\frac{u(x) - u(x_0)}{x - x_0}\right| \leq M \quad \text{if } |x - x_0| \quad \text{is sufficiently small,} \tag{2.8}$$

and

$$\left|\frac{u(x) - u(x_1)}{x - x_1}\right| \leq M \quad \text{if } |x - x_1| \quad \text{is sufficiently small.} \tag{2.9}$$

Given $\delta > 0$ sufficiently small choose $j = j(\delta)$ sufficiently large so that

$$\|u_j - u\|_{C([a,b])} \leq \delta. \tag{2.10}$$

For this j define

$$u_\delta(x) = \begin{cases} u_j(x) & \text{if } a \leq x \leq x_0 - \delta \\ \delta^{-1}[(x - x_0 + \delta)u(x_0) + (x_0 - x)u_j(x_0 - \delta)] & \text{if } x_0 - \delta \leq x \leq x_0 \\ u(x) & \text{if } x_0 \leq x \leq c \\ v(x) & \text{if } c \leq x \leq d \\ u(x) & \text{if } d \leq x \leq x_1 \\ \delta^{-1}[(x_1 + \delta - x)u(x_1) + (x - x_1)u_j(x_1 + \delta)] & \text{if } x_1 \leq x \leq x_1 + \delta \\ u_j(x) & \text{if } x_1 + \delta \leq x \leq b. \end{cases} \tag{2.11}$$

Since u_j is a minimizer, $I_{(a,b)}(u_\delta) \geq I_{(a,b)}(u_j)$, and hence

$$I_{(x_0-\delta,x_0)}(u_\delta) + I_{(x_0,c)}(u) + I_{(c,d)}(v) + I_{(d,x_1)}(u) + I_{(x_1,x_1+\delta)}(u_\delta) \geq I_{(x_0-\delta,x_1+\delta)}(u_j). \tag{2.12}$$

Now for $x \in (x_0 - \delta, x_0)$

$$\begin{aligned} |u_\delta'(x)| &= \delta^{-1}|u(x_0) - u_j(x_0 - \delta)| \\ &\leq \delta^{-1}|u(x_0) - u(x_0 - \delta)| + \delta^{-1}|u(x_0 - \delta) - u_j(x_0 - \delta)| \\ &\leq M + 1, \end{aligned} \tag{2.13}$$

and so $f(x, u_\delta(x), u_\delta'(x))$ is bounded on $(x_0 - \delta, x_0)$ independently of δ. Hence

$$\lim_{\delta \to 0} I_{(x_0-\delta,x_0)}(u_\delta) = 0, \tag{2.14}$$

and similarly

$$\lim_{\delta \to 0} I_{(x_1,x_1+\delta)}(u_\delta) = 0. \tag{2.15}$$

Also, since $f \geq 0$ and by weak lower semicontinuity,

$$\begin{aligned} \liminf_{\delta \to 0} I_{(x_0-\delta,x_1+\delta)}(u_{j(\delta)}) &\geq \liminf_{\delta \to 0} I_{(x_0,x_1)}(u_{j(\delta)}) \\ &\geq I_{(x_0,x_1)}(u). \end{aligned} \tag{2.16}$$

The weak lower semicontinuity implies also that

$$I_{(x_0,c)}(u) < \infty, \quad I_{(d,x_1)}(u) < \infty, \tag{2.17}$$

and so combining (2.12), (2.14)–(2.17) we obtain $I_{(c,d)}(v) \geq I_{(c,d)}(u)$, as required.

Lemma 2.2. *Let $a < b$ and let $u \in W^{1,1}(a,b)$ be a minimizer of I on $[c,b]$ for any $c \in (a,b)$. If there exists a minimizer u_0 of I on $[a,b]$ with $u_0(a) = u(a)$, $u_0(b) = u(b)$ and $|u_0'(a)| < \infty$, then u also minimizes I on $[a,b]$.*

Proof. For $\varepsilon > 0$ sufficiently small let $\delta(\varepsilon) \in (0, \frac{\varepsilon}{2})$ be such that

$$|u(a+\delta(\varepsilon)) - u(a)| \leq \varepsilon, \tag{2.18}$$

and define

$$u_\varepsilon(x) = \begin{cases} (\varepsilon - \delta(\varepsilon))^{-1}[(x - a - \delta(\varepsilon))\, u_0(a+\varepsilon) + (a + \varepsilon - x)u(a+\delta(\varepsilon))] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } a + \delta(\varepsilon) \leq x \leq a + \varepsilon \\ u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } \ a + \varepsilon \leq x \leq b. \end{cases} \tag{2.19}$$

Then $u_\varepsilon(a + \delta(\varepsilon)) = u(a + \delta(\varepsilon))$ and so

$$I_{(a+\delta(\varepsilon),b)}(u_\varepsilon) \geq I_{(a+\delta(\varepsilon),b)}(u). \tag{2.20}$$

It is easily verified that $|u_\varepsilon'(x)| \leq M < \infty$ on $(a + \delta(\varepsilon), a + \varepsilon)$, and so letting $\varepsilon \to 0$ we obtain $I_{(a,b)}(u) \leq I_{(a,b)}(u_0)$ as required.

Given $\varepsilon > 0$, $\rho > 0$ we denote by $R^\rho_{+,\varepsilon}$ (respectively $R^\rho_{-,\varepsilon}$) the subset of $\mathscr{D}_f$ consisting of those points $(a, \alpha) \in \mathbb{R}^2$ such that whenever $0 < \varepsilon' < \varepsilon$ there exists a minimizer u of I on $[a, a+\varepsilon']$ with $u(a) = \alpha$, $u'(a) = +\infty$ (respectively $-\infty$) and $I_{(a,a+\varepsilon')}(u) \leq \rho$. Similarly, we denote by $L^\rho_{+,\varepsilon}$ (respectively $L^\rho_{-,\varepsilon}$) the set of points $(a, \alpha) \in \mathbb{R}^2$ such that whenever $0 < \varepsilon' < \varepsilon$ there exists a minimizer u of I on $[a - \varepsilon', a]$ with $u(a) = \alpha$, $u'(a) = +\infty$ (respectively $-\infty$) and $I_{(a-\varepsilon',a)}(u) \leq \rho$. We set

$$R^\rho_\varepsilon := R^\rho_{+,\varepsilon} \cup R^\rho_{-,\varepsilon},$$

$$L^\rho_\varepsilon := L^\rho_{+,\varepsilon} \cup L^\rho_{-,\varepsilon}.$$

We will use the fact that R^ρ_ε is the same as the set $\hat{R}^\rho_\varepsilon$ consisting of those (a, α) such that whenever $0 < \varepsilon' < \varepsilon$ there exists a minimizer u of I on $[a, a+\varepsilon']$ with $u(a) = \alpha$, $|u'(a)| = \infty$ and $I_{(a,a+\varepsilon')}(u) \leq \rho$. Clearly $R^\rho_\varepsilon \subset \hat{R}^\rho_\varepsilon$. Conversely, if $(a, \alpha) \in \hat{R}^\rho_\varepsilon$ then there exists a sequence $\varepsilon_j \to \varepsilon$, $0 < \varepsilon_j < \varepsilon$, such that the corresponding minimizers u_j of I on $[a, a+\varepsilon_j']$ satisfy either $u_j'(a) = +\infty$ for all j or $u_j'(a) = -\infty$ for all j. Since u_j is a minimizer of I on $[a, a+\varepsilon']$ for $0 < \varepsilon' \leq \varepsilon_j$ and $I_{(a,a+\varepsilon')}(u_j) \leq I_{(a,a+\varepsilon_j)}(u_j)$ it thus follows that $(a, \alpha) \in R^\rho_{+,\varepsilon} \cup R^\rho_{-,\varepsilon}$. Hence $R^\rho_\varepsilon = \hat{R}^\rho_\varepsilon$. With the obvious definition we also have that $L^\rho_\varepsilon = \hat{L}^\rho_\varepsilon$.

A proof of the next lemma can be found in Ball and Mizel [1].

Lemma 2.3. **(Tonelli [7, II, p. 344 ff])** *Let $k > 0,\ m > 0,\ r > 0,\ M > 0$. Then there exists $\delta > 0$ such that if $|x_0| \le k$, $0 < x_1 - x_0 \le \delta$, $|u_0| \le m$ and*

$$\left|\frac{u_1 - u_0}{x_1 - x_0}\right| \le M$$

there is a unique solution $u \in C^2([x_0, x_1])$ of the Euler-Lagrange equation

$$\frac{d}{dx} f_p = f_u \tag{2.21}$$

satisfying $u(x_0) = u_0,\ u(x_1) = u_1$ and $|u(x) - u_0| \le r$ for all $x \in [x_0, x_1]$.

Furthermore u is the unique absolute minimizer of $I_{(x_0,x_1)}(v)$ in the set

$$\mathscr{A} = \{v \in W^{1,1}(x_0, x_1) : v(x_0) = u_0,\ v(x_1) = u_1,\ \max_{x \in [x_0, x_1]} |v(x) - u_0| \le r\}$$

Lemma 2.4. *For each $\varepsilon > 0, \rho > 0$ the sets $R^\rho_\varepsilon, L^\rho_\varepsilon$ are closed.*

Proof. We prove the result for R^ρ_ε; the proof for L^ρ_ε is similar. Let $(a_n, \alpha_n) \in R^\rho_\varepsilon, n = 1, 2, \ldots$ with $(a_n, \alpha_n) \to (a, \alpha)$. We show that $(a, \alpha) \in \hat{R}^\rho_\varepsilon$. Let $0 < \varepsilon' < \varepsilon$ and pick $\varepsilon''\varepsilon'''$ with $\varepsilon' < \varepsilon'' < \varepsilon''' < \varepsilon$. By assumption there exists a minimizer u_n of I on $[a_n, a_n + \varepsilon''']$ with $u_n(a_n) = \alpha_n$, $|u_n'(a_n)| = \infty$ and $I_{(a_n, a_n+\varepsilon''')}(u_n) \le \rho$. Defining $u_n(x) = \alpha_n$ for $x < a_n$ we deduce from the last inequality and the superlinear growth condition that a subsequence, again denoted u_n, converges weakly in $W^{1,1}(a - 1, a + \varepsilon'')$, and thus uniformly in $[a - 1, a + \varepsilon'']$, to a function $u \in W^{1,1}(a - 1, a + \varepsilon'')$. By Lemma 2.1 (applied to any compact subinterval of $(a, a + \varepsilon'')$) u is a minimizer of I on [c,d] whenever $a < c < d < a + \varepsilon''$ and by weak lower semicontinuity $I_{(a,a+\varepsilon'')}(u) \le \rho$.

Let $\tilde{u}$ be a minimizer of $I_{(a,a+\varepsilon')}(v)$ on the set

$$\mathscr{A}' = \{v \in W^{1,1}(a, a + \varepsilon') : v(a) = \alpha, v(a + \varepsilon') = u(a + \varepsilon')\}.$$

Since $u \in \mathscr{A}'$ we have $I_{(a,a+\varepsilon')}(\tilde{u}) \le \rho$. If $|\tilde{u}'(a)| = \infty$ we have thus found a suitable minimizer of I on $[a, a + \varepsilon']$. So suppose that $|\tilde{u}'(a)| < \infty$. Then by Lemma 2.2 u minimizes I on $[a, a + \varepsilon']$. If $|u'(a)| < \infty$ then there exists $M > 0$ such that

$$\left|\frac{u(b) - \alpha}{b - a}\right| \le \frac{1}{2} M$$

for all $b \in (a, a + \varepsilon')$. Let $k = \sup|a_n|, m = \sup\|u_n\|_{C([a_n, a_n+\varepsilon''])}$, $r = 2m$, and let $\delta > 0$ be given by Lemma 2.3. Choosing $x_0 = a_n$, $x_1 = a_n + \tau$, $0 < \tau < \min(\delta, \varepsilon')$, and noting that

$$\lim_{n\to\infty} \left|\frac{u_n(a_n + \tau) - u_n(a_n)}{\tau}\right| \le \frac{1}{2} M,$$

we deduce that for n sufficiently large $u_n \in C^2([a_n, a_n + \tau])$. This contradicts $|u_n'(a_n)| = \infty$ and hence $|u'(a)| = \infty$, so that u is a suitable minimizer on $[a, \alpha + \varepsilon']$. Since $\varepsilon' \in (0, \varepsilon)$ was arbitrary we have shown that $(a, \alpha) \in \hat{R}^\rho_\varepsilon$ as required. Hence $R^\rho_\varepsilon = \hat{R}^\rho_\varepsilon$ is closed.

Define

$$\begin{aligned} K_1 = &\ \{(x,y) \in \mathbb{R}^2 &: x \geq 0, y \geq 0\}, \\ K_2 = &\ \{(x,y) \in \mathbb{R}^2 &: x \geq 0, y \leq 0\}, \\ K_3 = &\ \{(x,y) \in \mathbb{R}^2 &: x \leq 0, y \geq 0\}, \\ K_4 = &\ \{(x,y) \in \mathbb{R}^2 &: x \leq 0, y \leq 0\}. \end{aligned}$$

We say that a set $E \subset \mathbb{R}^2$ is K_i*-closed* if any point $z \in \mathbb{R}^2$ such that $z \in \overline{E \cap (K_i + z)}$ is in E.

Lemma 2.5. *Let* $\varepsilon > 0$, $\rho > 0$. *Then* $R^{\rho}_{+,\varepsilon}$ *is* K_4*-closed,* $R^{\rho}_{-,\varepsilon}$ *is* K_3*-closed,* $L^{\rho}_{+,\varepsilon}$ *is* K_2*-closed, and* $L^{\rho}_{-,\varepsilon}$ *is* K_1*-closed.*

Proof. We show that $R^{\rho}_{+,\varepsilon}$ is K_4-closed; the proofs of the other assertions are similar. Let $z = (a,\alpha)$ and $(a_n,\alpha_n) \in R^{\rho}_{+,\varepsilon} \cap (K_4 + z)$ with $(a_n,\alpha_n) \to (a,\alpha)$ as $n \to \infty$. Let $0 < \varepsilon' < \varepsilon'' < \varepsilon''' < \varepsilon$. By assumption there exists a minimizer u_n of I on $[a_n, a_n + \varepsilon''']$ with $u_n(a_n) = \alpha_n$, $u_n'(a_n) = +\infty$ and $I_{(a_n,a_n+\varepsilon''')}(u_n) \leq \rho$. As in the proof of Lemma 2.4 we may suppose that $u_n \to u$ in $W^{1,1}(a-1, a+\varepsilon'')$, where u is a minimizer of I on [c,d] whenever $a < c < d < a + \varepsilon''$ and where $I_{(a,a+\varepsilon'')}(u) \leq \rho$. Since $u \in W^{1,1}(a-1, a+\varepsilon'')$, u is differentiable at some point $a + \hat{\varepsilon}$ with $\varepsilon' < \hat{\varepsilon} < \varepsilon''$. Let $\tilde{u}$ be a minimizer of $I_{(a,a+\hat{\varepsilon})}(v)$ on the set $\hat{\mathscr{A}} = \{v \in W^{1,1}(a, a+\hat{\varepsilon}) : v(a) = \alpha, v(a+\hat{\varepsilon}) = u(a+\hat{\varepsilon})\}$.

Suppose $\tilde{u}'(a) \neq +\infty$. We first show that in this case u is a minimizer of I on $[a, a+\hat{\varepsilon}]$. This follows as in Lemma 2.4 if $|\tilde{u}'(a)| < \infty$. If $\tilde{u}'(a) = -\infty$ we argue as follows. Since $a_n \leq a$, $\alpha_n \leq \alpha$ there exists for large enough n points $x_{1n} \geq a$ with $\tilde{u}(x_{1n}) = \alpha_n$ and $\lim_{n\to\infty} x_{1n} = a$. Also, since $u_n \to u$ uniformly in $[a-1, a+\varepsilon'']$ and u is differentiable at $a+\hat{\varepsilon}$ there exists a sequence $\delta_n > 0$ with $\lim_{n\to\infty} \delta_n = 0$ such that

$$\left| \frac{u_n(a+\hat{\varepsilon}+\delta_n) - u(a+\hat{\varepsilon})}{\delta_n} \right| \leq C < \infty \tag{2.22}$$

for all n. Now define

$$\tilde{u}_n(x) = \begin{cases} \alpha_n & \text{if } a_n \leq x \leq x_{1n} \\ \tilde{u}(x) & \text{if } x_{1n} < x \leq a+\hat{\varepsilon} \\ u(a+\hat{\varepsilon}) + (x-a-\hat{\varepsilon})\delta_n{}^{-1}(u_n(a+\hat{\varepsilon}+\delta_n) & \\ -u(a+\hat{\varepsilon})) & \text{if } a+\hat{\varepsilon} < x \leq a+\hat{\varepsilon}+\delta_n. \end{cases}$$

Since u_n is a minimizer of I on $[a_n, a+\hat{\varepsilon}+\delta_n]$ for sufficiently large n, it follows that

$$I_{(a_n,a+\hat{\varepsilon}+\delta_n)}(\tilde{u}_n) \geq I_{(a_n,a+\hat{\varepsilon}+\delta_n)}(u_n). \tag{2.23}$$

Using (2.22) we see that

$$\lim_{n\to\infty} I_{(a_n,a+\hat{\varepsilon}+\delta_n)}(\tilde{u}_n) = I_{(a,a+\hat{\varepsilon})}(\tilde{u}), \tag{2.24}$$

while by weak lower semicontinuity and the positivity of f

$$\begin{aligned} \liminf_{n\to\infty} I_{(a_n,a+\hat{\varepsilon}+\delta_n)}(u_n) &\geq \liminf_{n\to\infty} I_{(a,a+\hat{\varepsilon})}(u_n) \\ &\geq I_{(a,a+\hat{\varepsilon})}(u). \end{aligned} \tag{2.25}$$

Combining (2.23)–(2.25) we deduce that

$$I_{(a,a+\hat{\varepsilon})}(\tilde{u}) \geq I_{(a,a+\hat{\varepsilon})}(u), \tag{2.26}$$

so that u is a minimizer of I on $[a, a+\hat{\varepsilon}]$ as claimed.

Using Lemma 2.3 as in the proof of Lemma 2.4 we see that $|u'(a)| = \infty$. Suppose that $u'(a) = -\infty$. Then since $a_n \leq a$, $\alpha_n \leq \alpha$, $u_n'(a_n) = +\infty$ and $u_n \to u$ uniformly in $[a-1, a+\varepsilon'']$ it follows that for sufficiently large n there exist $y_n > a_n$ with $u_n(y_n) = \alpha_n$ and $\lim_{n\to\infty} y_n = a$. Since the u_n are uniformly bounded and $y_n - a_n \to 0$ it follows from Lemma 2.3 that $u_n \in C^2([a_n, y_n])$, contradicting $u_n'(a_n) = +\infty$.

From the above arguments we see that either $\tilde{u}(a_n) = +\infty$ or $u'(a) = +\infty$ and u is a minimizer on $[a, a+\hat{\varepsilon}]$. In both cases we have a minimizer on $[a, a+\varepsilon']$ with derivative $+\infty$ at a. Thus $z \in R^{\rho}_{+,\varepsilon}$ and hence $R^{\rho}_{+,\varepsilon}$ is K_4-closed.

Proof of Theorem 2.1. Since $\mathscr{D}_f = \cup_{i=1}^{\infty} R^i_{1/i} \cup L^i_{1/i}$, and since by Lemma 2.4 all the sets $R^i_{1/i}, L^i_{1/i}$ are closed, it suffices to prove that each of the sets $R^i_{1/i}, L^i_{1/i}$ is nowhere dense. Let us suppose the contrary, that the set $R^j_{1/j}$, say, contains a nonempty open set G. Since $R^j_{1/j} = R^j_{+,1/j} \cup R^j_{-,1/j}$ it follows that one of the sets $R^j_{+,1/j}, R^j_{-,1/j}$ is dense in an open subset Ω of G, say $\overline{R^j_{+,1/j}} \supset \Omega$. Then for any $z \in \Omega$ the set $R^j_{+,1/j} \cap (K_4 + z)$ is dense in $\Omega \cap (K_4 + z)$, and hence by Lemma 2.5 $z \in R^j_{+,1/j}$ and $\Omega \subset R^j_{+,1/j}$.

Set

$$Q_h = \{(x,y) \in \mathbb{R}^2 : |x| < h,\ |y| < h\}.$$

Choose $z \in \Omega$ and a positive $h < \frac{1}{2j}$ such that $Q_h + z \subset \Omega$. Write $Q = Q_h + z$, $z = (a, \alpha)$, and let $x_1 = a-h,\ x_2 = a+h$. Since $\varphi(p)/|p| \to \infty$ as $|p| \to \infty$ there exists a constant $K > 0$ such that $\varphi(p) \geq |p| - K$ for all p. Choose $l > (2K+1)h + j$ and let $y_1 = \alpha - l,\ y_2 = \alpha + l$. Let u be a minimizer of I on $[x_1, x_2]$ with $u(x_1) = y_1,\ u(x_2) = y_2$. Then there exists a point $(b, \beta) \in Q$ such that $u(b) = \beta$ and $|u'(b)| < \infty$. By assumption there exists a minimizer $\tilde{u}$ on $[b, x_2]$ with $\tilde{u}(b) = \beta,\ \tilde{u}'(b) = +\infty$ and $I_{(b,x_2)}(\tilde{u}) \leq j$. Now

$$\begin{aligned} |\tilde{u}(x_2) - \tilde{u}(b)| &\leq \int_b^{x_2} |\tilde{u}'|\, dx \\ &\leq \int_b^{x_2} [K + \varphi(\tilde{u}')]\, dx \\ &\leq K(x_2 - b) + j, \end{aligned}$$

and hence $\tilde{u}(x_2) \leq \alpha + h + 2Kh + j < y_2$. Thus there exists a point $c \in (b, x_2)$ with $u(c) = \tilde{u}(c)$. Set

$$v(x) = \begin{cases} u(x) & \text{if } x_1 \leq x \leq b \\ \tilde{u}(x) & \text{if } b < x \leq c. \end{cases}$$

Then v is a minimizer of I on $[x_1, c]$. But v has no derivative at $x = b$, either finite or infinite, contradicting Tonelli's result.

Proof of Theorem 2.2. Let $(a, \alpha) \in \mathscr{D}_{f,E,M}$, and let u be a minimizer on E with $u(a) = \alpha$, $|u'(a)| = \infty$, $|u(A)| \leq M$ and $|u(B)| \leq M$. Denote by $C_{M,A,B}$ the maximum of f on $[A, B] \times [-M, M] \times [\frac{-2M}{B-A}, \frac{2M}{B-A}]$. Letting $l_u(x) = u(A) + (\frac{u(B)-u(A)}{B-A})(x - A)$, we see that

$$I(u) \leq I(l_u) \leq (B - A) C_{M,A,B}.$$

Pick $j > \max\{(B-A)C_{M,A,B}, \frac{2}{B-A}\}$. Then $(a, \alpha) \in R^j_{1/j} \cup L^j_{1/j}$, and so $\mathscr{D}_{f,E,M} \subset R^j_{1/j} \cup L^j_{1/j}$. But as shown in the proof of Theorem 2.1 the set $R^j_{1/j} \cup L^j_{1/j}$ is closed and nowhere dense.

3 Discussion

It would be useful to determine whether $\mathscr{D}_f$ has two-dimensional Lebesgue measure zero, and if so to calculate its Hausdorff dimension, but we have not succeeded in these tasks. It is not even clear to us whether or not the projection of $\mathscr{D}_f$ onto the x-axis has zero one-dimensional Lebesgue measure. In this connection we recall the example in Ball and Mizel [1] of a smooth nonnegative integrand $f = f(u,p)$ satisfying $f_{pp} > 0$ and $f(u,p)/|p| \to \infty$ as $|p| \to \infty$ for each $u \neq 0$, but for which the Tonelli set E is nonempty. Since this integrand does not depend on x it follows that the projection of $\mathscr{D}_f$ onto the x-axis is the entire x-axis. However this f does not satisfy the growth hypothesis (1.2).

An interesting question is whether under suitable hypotheses small universal singular sets exist for multiple integrals

$$I(u) = \int_\Omega f(x,u,Du)\,\mathrm{d}x \tag{3.1}$$

depending on mappings $u : \Omega \to \mathbb{R}^N$, where $\Omega \subset \mathbb{R}^n$ is bounded and open. In this context we recall the example of Nečas [4] for which $N = n^2$ with n sufficiently large, and $f = f(Du)$ is analytic and satisfies the uniform convexity condition

$$D^2 f(A)(\xi,\xi) \geq c|\xi|^2 \quad \text{for all} \quad A,\xi\ , \tag{3.2}$$

where $c > 0$. The corresponding Euler-Lagrange equation has as a weak solution the mapping u^0 given by

$$u^{0ij}(x) = \frac{x^i x^j}{|x|}\ , \tag{3.3}$$

which is Lipschitz, but is not differentiable at the origin. Since f depends only on Du, given any constant vectors $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^N$ the mapping $u^0_{a,\alpha}$ defined by

$$u^0_{a,\alpha}(x) = u^0(x-a) + \alpha \tag{3.4}$$

is also a weak solution, and since f is strictly convex it follows that $u^0_{a,\alpha}$ is the unique absolute minimizer of $I(u)$ subject to the boundary condition $u|_{\partial\Omega} = u^0_{a,\alpha}$. Hence for any $a \in \Omega$, $\alpha \in \mathbb{R}^N$ there exists a minimizer $u^0_{a,\alpha}$ which is not smooth at the point a, with $u^0_{a,\alpha}(a) = \alpha$, in contrast to our one-dimensional result.

However, this example is not a fair comparison since in one dimension Lipschitz minimizers are necessarily smooth. A fairer comparison would be to consider smooth integrands $f : \Omega \times \mathbb{R}^N \times M^{m\times n} \to \mathbb{R}$, with $f(x,u,\cdot)$ strictly convex (or, more generally, strictly quasi-convex), where $M^{m\times n}$ denotes the set of real $m\times n$ matrices, satisfying the growth condition

$$f(x,u,A) \geq c_0|A|^p + c_1 \tag{3.5}$$

for constants $p > n$, $c_0 > 0$ and c_1. The growth condition (3.5) ensures that any minimizer of I on Ω is continuous, and we can ask whether the set

$$\mathscr{D}_{f,\Omega} = \{(a,\alpha) \in \Omega \times \mathbb{R}^N \ :\ \text{there exists a minimizer } u \text{ of } I \text{ on } \Omega \text{ with } u(a) = \alpha \text{ and } Du \text{ unbounded in any neighbourhood of } a\}$$

is in some sense small. As far as the authors are aware there is no counterexample to this in the literature. In particular there is no example known of an integrand $f = f(Du)$ satisfying (3.5) with a corresponding minimizer having Du unbounded at an interior point of Ω.

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