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Universität Münster
Wintersemester 2022/23

Geometric Group Theory I
Exercise Sheet 4

Exercise 1.

a) Suppose X and Y are subtrees of a tree T. Show that if $X \cap Y$ is non-empty, then it is a subtree of T.

b) Let T be a tree and $Y \subseteq T$ be a union of disjoint subtrees in T. Consider the graph T/Y obtained by contracting the trees in Y.

   i) Let X be a subtree of T, show that there is a natural injective morphism $\iota_X : X/(X \cap Y) \to T/Y$.

   ii) Let $\{T_i : i \in I\}$ be the directed system of all finite subtrees of T. Show that $\{T_i/(T_i \cap Y) : i \in I\}$ form a directed system of subtrees.

   iii) Show that the map $\iota : \bigcup_{i \in I} T_i/(T_i \cap Y) \to T/Y$ is an isomorphism of graphs, where $\iota := \bigcup_{i \in I} \iota_{T_i}$, namely ι coincides with $\iota_{T_i} : T_i/(T_i \cap Y) \to T/Y$ when restricted on each $T_i/(T_i \cap Y)$.

(4 Points)

Let G be the free group with basis S and let H be a subgroup. A *Schreier transversal* for H in G is a set $\mathcal{T}$ of reduced words in S such that each right coset of H in G contains a unique word of $\mathcal{T}$ and **all** initial segments of these words also lie in $\mathcal{T}$.

Exercise 2.

a) Describe a Schreier transversal for the commutator subgroup H of $F(a, b) := F(\{a, b\})$. *Hint: Note that $F(a, b)/H$ is abelian.*

b) Let H be the kernel of the homomorphism $\varphi : F(a, b) \to S_3$ defined by $\varphi(a) := (1\ 2)$ and $\varphi(b) := (1\ 3)$. Describe a Schreier transversal for H in $F(a, b)$.

(4 Points)

Exercise 3. Let G be the free group with basis S. Let $\Gamma(G,S)$ be the Cayley graph. Recall that G acts naturally on $\Gamma(G,S)$ by $h*(g,s) := (hg,s)$ for any $h \in G$ and $(g,s) \in \Gamma(G,S)^+$. Let H be a subgroup of G. Consider the quotient graph $Y := H\backslash\Gamma(G,S)$. Define the *label map* $\gamma : Y^1 \to S \cup \{S^{-1}\}$ as: $\gamma((Hg,s)) := s$ and $\gamma(\overline{(Hg,s)}) := s^{-1}$ for $(Hg,s) \in Y^+$. For a path $p = (e_0, \dots, e_{n-1})$ in Y, define the label $\gamma(p) := \prod_{i<n} \gamma(e_i) \in G$ if $n > 0$ and $\gamma(p) := 1 \in G$ if $n = 0$.

Let Δ be a maximal subtree of Y containing the vertex $H \in Y^0$. For any $v \in \Delta^0$, let p_v be the geodesic from H to v in Δ. Show that: the set

$$\mathcal{T}(\Delta) := \{\gamma(p_v) : v \in \Delta^0\}$$

is a Schreier transversal for H in G. Conclude that any subgroup of a free group G has a Schreier transversal in G.

(4 Points)

Exercise 4.

a) Let $G = H \rtimes D$ be a semidirect product, and let the subgroups H and D have presentations $\langle X \mid R \rangle$ and $\langle Y \mid S \rangle$ respectively. Show that G has the presentation:

$$\langle X \cup Y \mid R \cup S \cup \{y^{-1}xy = w_{xy} : x \in X, y \in Y^{\pm}\} \rangle,$$

where w_{xy} is a reduced word in X, representing the element $y^{-1}xy$ of H.

b) Use part a) of this exercise to find a presentation of the finite dihedral group D_n.

(4 Points)

*Submission by **Wednesday** morning 11:00, 09.11.2022, in Briefkasten 161.*
The exercise sheets should be solved and submitted in pairs.
Tutorial: Fridays 12:00-14:00, in room SR1d.
If you have questions about the problem sheet, please write to Tingxiang: tingxiangzou@gmail.com.