# A Robust(ish) Scott Rank for Separable Metric Structures

Special thanks to Dino Rossegger for all the productive conversations

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Geometry from the model theorist's point of view

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We say that a metric structure is **separable** if $(M, d)$ is a separable metric space and $I$ is countable.



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$$(M, d_{\{0,1\}}; \{c_j\}_{j\in J}, \{f_k\}_{k\in K}, \{R_\ell\}_{\ell\in L})$$

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is a metric structure.
2. Cantor and Baire space: $(2^\omega, d)$ and $(\omega^\omega, d)$ where
$$d(\tau,\sigma) = \begin{cases} 2^{-n}, & \text{if } n \text{ is the least index such that } \tau(n) \neq \sigma(n) \\ 0, & \tau = \sigma \end{cases}$$
We can also add a function $f : M \to M$ defined by $f(\tau)(n) = \tau(n+1)$.

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$$\mathcal{A} = (A, 0, 1, \neg, \wedge, \vee, \mu)$$

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2. Hilbert Space.

# Metric Scott analysis

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## Theorem ([BYDNT17] Scott Sentences)

*Every separable metric structure $\mathcal{A}$ is characterized, up to isomorphism among all separable metric structures in the same language, by a continuous infinitary sentence, which is called the Scott sentence of $\mathcal{A}$. That is, there is a continuous infinitary sentence $\phi$ such that for any separable metric structure $\mathcal{B}$ in the same language,*

$$\phi^{\mathcal{B}} = 0 \quad \textit{iff} \quad \mathcal{B} \cong \mathcal{A}.$$



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3. $\sup_x$ and $\inf_x$ act as the continuous first-order quantifiers;
4. Suppose $\{\phi_n \mid n < \omega\}$ is a set of formulas, $\Delta$ a modulus of continuity, and $I \subset \mathbb{R}$ compact. If each $\phi_n$ respects $\Delta$ and $I$, then $\sup_n \phi_n$ and $\inf_n \phi_n$ are formulas

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## Theorem ([Mon15])

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3. *for any $\Pi^{in}_{\alpha}$-type $p(\bar{x})$ realized in $\mathcal{A}$, there is a $\Sigma^{in}_{\alpha}$ formula $\phi(\bar{x})$ that supports the type in $\mathcal{A}$. Meaning that $\mathcal{A} \models \exists\, \bar{x}\phi(\bar{x})$ and:*

$$\mathcal{A} \models \forall x \left(\phi(\bar{x}) \Rightarrow \bigwedge_{\psi(x)\in p(x)} \psi(\bar{x})\right).$$



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## Question

*Can we give a robust notion of Scott rank for separable metric structures?*

# A First Step

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## Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be separable metric structures. Fix sequences $A = \{a_n \mid n < \omega\} \subset \mathcal{A}$ and $B = \{b_n \mid n < \omega\} \subset \mathcal{B}$ such that every tail of each sequence is dense in the respective structure.



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## Question

*What relation between $A$ and $B$ can be lifted to an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.*

# Definition

An $\Omega$*-bounded back-and-forth set with bound* $t > 0$ is a set $I \subset A^{<\omega} \times B^{<\omega}$ such that for $(\overline{a}, \overline{b}) \in I$ we have:

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An $\Omega$*-bounded back-and-forth set with bound t* $> 0$ is a set $I \subset A^{<\omega} \times B^{<\omega}$ such that for $(\bar{a}, \bar{b}) \in I$ we have:

1. $\sup_\phi |\phi^{\mathcal{A}}(\bar{a}) - \phi^{\mathcal{B}}(\bar{b})| < t$ where $\phi(\bar{x})$ varies over all $|\bar{a}| = |\bar{b}|$-ary $\Omega$-quantifier free formulas*;

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2. For every $c \in A$ there is a $d \in B$ such that the index of $d$ in $B$ is larger than all the indices in $\bar{a}, \bar{b}, c$ and $(\bar{a}c, \bar{b}d) \in I$;
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## Theorem

*Let $\mathcal{A}$ and $\mathcal{B}$ be separable metric structures, $\Omega$ a universal modulus, and $t > 0$. Fix countable tail-dense sequences A of $\mathcal{A}$ and B of $\mathcal{B}$. If $I \subset A^{<\omega} \times B^{<\omega}$ is an $\Omega$-bounded back-and-forth set with bound t and $(\overline{a}, \overline{b}) \in I$, then there is an isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ such that*

$$\Omega(d_{\mathcal{A}}(f(a_i), b_i) \mid i < |\overline{a}|) \leq t.$$



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## Lemma ([BYDNT17])

*Let $\Omega$ be an universal weak-modulus, $\mathcal{A}$ be a separable metric structure with a countable tail-dense sequence A, and $\overline{a}, \overline{b} \in A^n$ for some $n < \omega$. Then $\overline{b} \in \overline{Aut_{\mathcal{A}}(\overline{a})}$ if, and only if, $\phi^{\mathcal{A}}(\overline{a}) = \phi^{\mathcal{A}}(\overline{b})$ for all $\Omega$ continuous infinitary formulas without parameters.*

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## Theorem (B.)

*Let $\Omega$ be an universal weak-modulus, $\mathcal{A}$ be a separable metric structure. Then the closure of the automorphism orbit of $\bar{a} \in \mathcal{A}^{<\omega}$ in $\mathcal{A}$ is $\Omega$-definable (i.e. the function $d^{\Omega}(x, \overline{Aut_{\mathcal{A}}(\bar{a})})$ is a definable predicate).*



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## Definition

The $\Omega$-Scott Rank of a separable metric structure $\mathcal{A}$ is the least countable ordinal $\alpha > 0$ such that all the automorphism orbits of all finite tuples of $\mathcal{A}$ are $(\Omega, \mathbf{inf}^{\alpha})$-definable.



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## Theorem (B.)

*Let $\mathcal{A}$ be a separable metric structure , $\Omega$ a universal modulus and $\alpha > 0$ a countable limit ordinal. Fix A, a countable tail-dense sequence of $\mathcal{A}$. Then, the following are equivalent:*

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3. *every $(\Omega, \overline{\sup^{<\alpha}})$ type realized in A is supported in $\mathcal{A}$ by an $(\Omega, \inf^{<\alpha})$-definable predicate without parameters.*

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*Thank You!*