Pure morphisms

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Geometry from the model theorist's point of view
2024

Coherent categories

Definition

L is a finitary signature.
$L^g_{\omega\omega} = \{\forall \vec{x}(\varphi(\vec{x}) \to \psi(\vec{x})) : \varphi, \psi \text{ pos. ex.}\}$ (pos. ex.: atomic, $\wedge$, $\vee$, $\exists$)

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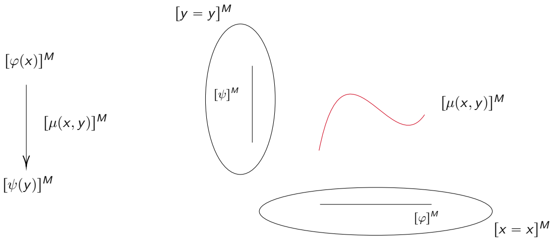
M is an L-structure $\rightsquigarrow$ $Def(M)$: category of (pos. ex.) definable sets and (pos. ex.) definable functions.



Observation: $Def(M)$ is closed under some universal constructions, e.g.:

- finite products: $([\varphi_i(\vec{x_i})]^M)_{i<n}$ their product is $[\bigwedge_i \varphi_i(\vec{x_i'})]^M$ (renamed variables).
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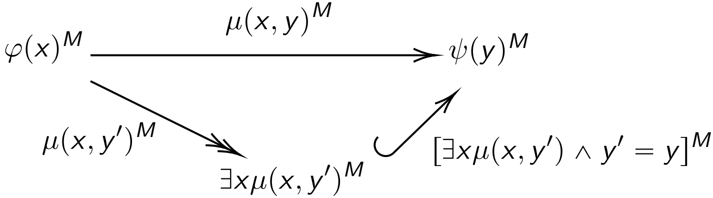

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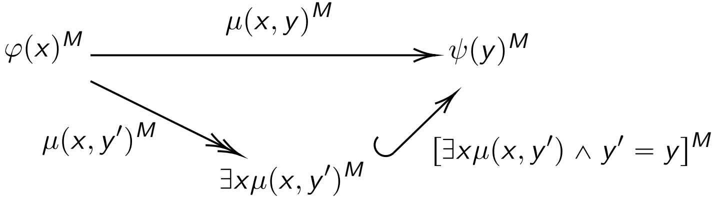

- Also: equalizers, finite unions.

Idea: *Def* (M) is the ev_M-image of some abstract "category of formulas", these constructions live there, ev_M preserves them.

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Conversely: every small coherent category encodes a many-sorted coherent theory $Th(\mathcal{C}) \subseteq (L_{\mathcal{C}})^g_{\omega\omega}$ (s.t. $Mod(Th(\mathcal{C})) = \mathbf{Coh}(\mathcal{C}, \mathbf{Set})$, etc).

Accessible categories

$Mod(T) = \mathbf{Coh}(\mathcal{C}, \mathbf{Set})$ has the following properties:

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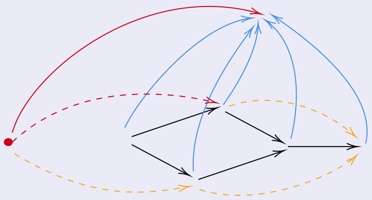
Claim: accessible categories are precisely the categories of the form $Mod(T)$ for some $T \subseteq L^g_{\mu\lambda}$.
(Like AECs except: λ-version of Tarski-Vaught chain axiom & maps are not monos.)

Definition

An object $x \in \mathcal{A}$ is λ-presentable if $\mathcal{A}(x,-)$ preserves λ-directed colimits.

Remark



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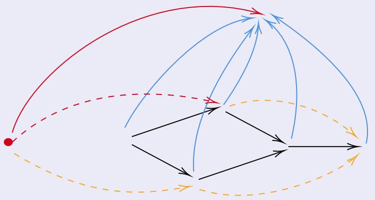

Example:
Set $\ni x$ is λ^+-presentable iff it has cardinality $\leqslant \lambda$
Ab $\ni x$ is $\aleph_0$-presentable iff it is finitely presentable, it is λ^+-presentable iff it has cardinality $\leqslant \lambda$.
In **Top** the Sierpiński-space is not λ-presentable for any λ.

Proposition

$T \subseteq L^{g}_{\mu\lambda} \rightsquigarrow Mod(T)$ is accessible with λ-directed colimits.
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Example

$|L| \leqslant \aleph_0$. Then for any $T \subseteq L^g_{\omega\omega}$: $Mod(T)$ is $\aleph_1$-accessible.

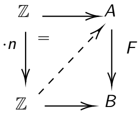
Example

1. **Ab** is $\aleph_0$-accessible.
2. Let L be countable and $T \subseteq L^g_{\omega\omega}$ be $\aleph_0$-categorical with no finite models. Then $Mod(T)$ is not $\aleph_0$-accessible.
3. Let $\mathcal{A}$ be the category of (directed, simple) graphs, satisfying $\forall x \exists y : R(x,y)$. It is not $\aleph_0$-accessible.

Is it possible to characterize theories $T \subseteq L^g_{\omega\omega}$ (say, over countable L) for which $Mod(T)$ is $\aleph_0$-accessible?

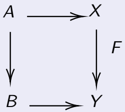
Pure maps

An injective map of Abelian groups $F : A \to B$ is pure if it reflects divisibility: $\exists x : F(a) = n \cdot x$ implies $\exists x : a = n \cdot x$.
This is the same as the following: if the square commutes then there is a lift, s.t. the upper triangle commutes.



Definition

$\mathcal{A}$ is λ-accessible. $F : X \to Y$ is λ-pure if for any A, B λ-presentable and comm. square

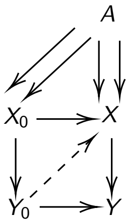


there is a lift $B \to X$ making the upper triangle commute.

Proposition

λ-pure $\Rightarrow$ monomorphism

Proof:



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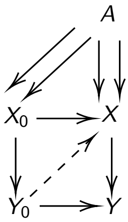

History:
[AR94]: "Is it true that λ-pure $\Rightarrow$ regular monomorphism?"
[AHT96]: "If $\mathcal{A}$ has pushouts: yes. In general: no."
[HP97]: "If $\mathcal{A}$ has products: yes."
goal: In $\mathbf{Coh}(\mathcal{C}, \mathbf{Set})$: yes (but with "strict" instead of "regular").

Immersions

A pure subgroup was: the validity of some pos. ex. formula is reflected.
Immersion: the validity of any pos. ex. formula is reflected.

Definition

$\mathcal{C}$ is lex, $F, G : \mathcal{C} \to \mathbf{Set}$ lex. $\alpha : F \Rightarrow G$ is elementary (or: immersion) if the naturality squares at monos are pullbacks.

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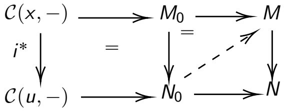
Proposition

If $\mathbf{Coh}(\mathcal{C}, \mathbf{Set})$ *is* λ*-accessible then* λ*-pure* $\Rightarrow$ *elementary.*

proof:

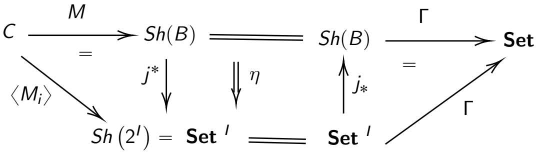


Idea: $\mathbf{Coh}(\mathcal{C}, \mathbf{Set}) \subseteq \mathbf{Lex}(\mathcal{C}, \mathbf{Set}) = Pro(\mathcal{C})^{op}$.

1. we know: λ-pure $\Rightarrow$ elementary.
2. claim: $\alpha : F \Rightarrow G$ is elementary iff regular mono in $\mathbf{Lex}(\mathcal{C}, \mathbf{Set})$.
3. then enough: every $F : \mathcal{C} \to \mathbf{Set}$ admits a (regular) mono to a product of coherent functors:

$$M \hookrightarrow N \rightrightarrows F \hookrightarrow \prod N_i$$

   then $M \to N$ is the joint equalizer of the $N \rightrightarrows N_i$ pairs in $\mathbf{Lex}(\mathcal{C}, \mathbf{Set})$ hence in $\mathbf{Coh}(\mathcal{C}, \mathbf{Set})$.
4. every lex embeds to regular: small object argument
5. every regular embeds to product of coherents [Lurie]:



References

[AR94] J. Adámek, J. Rosický: Locally presentable and accessible categories

[AHT96] J. Adámek, H. Hu, W. Tholen: On pure morphisms in accessible categories

[HP97] H. Hu, J. W. Pelletier: On regular monomorphisms in weakly locally presentable categories

[Lurie] J. Lurie: lecture notes in categorical logic

Pure maps are strict monomorphisms (arxiv.org/abs/2407.13448)