Parameters in AC Fields

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Definition of a bifield

- a structure definable in an ac field K

- $B = L_1 \cup L_2$, two copies of the base field K

- language: (i) the equivalence relation $x \in L_1 \leftrightarrow y \in L_2$

(ii) the graphs of the two field operations (the ternary relations $x +_1 y = z \vee x +_2 y = z$, $x \times_1 y = z \vee x \times_2 y = z$)

(iii) an automorphism $\tau = (\tau_1, \tau_2)$ of B , where τ_1 is a field-isomorphism from L_1 to L_2 and τ_2 a field-isomorphism from L_2 to L_1 , both definable in K .

Characteristic 0 : the identity is the only definable automorphism of K , τ_1 and τ_2 are inverses of each other and τ is an involution; the quotient of B by $u = v \vee u = \tau(v)$ is a third copy L_3 of K definable without parameters in B . No interest.

Characteristic p : the definable automorphisms of K are the maps $x \to x^{p^n}$, $n \in Z$. Since every y has the form $\tau_1(x)$, $\tau_2(\tau_1(x)) = x^{p^n}$ implies that $\tau_1(\tau_2(y)) = y^{p^n}$.

If $n = 2m$ is even, $\tau(x^{p^{-m}}, y^{p^{-m}})$ is an involutive automorphism of B : same situation as in characteristic 0 .

If n is odd, B has no involutive automorphism (cannot fix the fields by Artin's Theorem, cannot switch them because the Frobenius automorphism is not a square) and no copy of K is definable without parameters in B .

Definition of an autonomous constructible structure

S is infinite, definable in K, and every subset of a cartesian power of S which is definable in the sense of K is definable (with parameters) in the language of S.

Examples

- a multifield (same as a bifield, but with n fields)
- a simple algebraic group
- a quasi-simple algebraic group (connected, finite center, G/Z(G) is simple)

Basic Model Theory for AC fields

- elimination of quantifiers (definable = constructible)
- elimination of imaginaries
- any definable structure is pseudo locally finite
- any definable group is definably isomorphic to an algebraic group
- any definable infinite field is def. isomorphic to the base field
- any simple infinite algebraic group is definably isomorphic to a linear group definable without parameters
- a simple algebraic group is itself, in the group language, an ω_1-categorical structure (Zilber's Theorem)

Why the simple groups are autonomous?

In an autonomous structure $S = S(K)$, an infinite field L is definable, and if σ is an isomorphism from K to L definable in K , the induced isomorphism σ^* from $S(K)$ to $S(L)$ is definable in the language of S . All these definitions use parameters.

We must prove this for an algebraic simple group G ; the field L is defined in the borels of G , which are not nilpotent; in the group language, the generic of G is not orthogonal to L , and therefore G is L-internal by a theorem of Hrushovski.

Interpretation without parameters

In characteristic 0 , in any autonomous structure S a copy L of the base field K is interpretable without parameters.

Therefore any automorphism of S will induce an automorphism of the field L .

In characteristic p , in any autonomous structure one can interpret without parameters a multifield $(L_1, \ldots L_n)$ of copies of K .

Therefore, any automorphism of S whose action on the multifield has a finite order fixes pointwise some copy L of K definable in S .

Borel-Tits Theorem, model theoretic version

On isomorphisms. Let σ be an isomorphism between S and S', two autonomous constructible structures over ac field K and K' respectively; then there is an isomorphism between the field K and K', such that σ discomposes in the induced isomorphism between $S(K)$ and $S(K')$, followed by a map definable in S' (or in K' !).

On automorphisms. Let σ be an automorphism of an autonomous constructible structure S, L a field definable in S, and $S(L)$ a definable copy of S; then σ decomposes into a definable map, the induced isomorphism between $S(L)$ and $S(\sigma L)$, and a definable map. Therefore σ is definable (in S or in K, this is the same thing!) provided that it induces a definable map from L to σL.

Corollary: rigidity of autonomous structures

Let σ be an automorphism of an autonomous constructible structure S over an algebraically closed field.

(i) In characteristic 0, if σ has a finite order, σ^2 is constructible ; if (S,σ) is superstable, σ is constructible.

(ii) In characteristic p, if σ has a finite order, it is constructible ; if σ belongs to a superstable group of automorphisms of S, it is constructible.

More on simple algebraic groups

With the help of some algebraic geometry, one can say more while speaking of a simple algebraic group G .

Firs, one can define in G without parameters a copy of the base field, even in characteristic p .

Moreover, the constructible automorphisms of G mentionned in the last page are in fact algebraic morphisms. The group of all algebraic automorphisms of G is definable in K , its connected component being the group of inner automorphisms, which is isomorphic to G .

Therefore, if G is a normal subgroup of H such that (H,G) be superstable, the action of H° on G is by inner automorphisms of G : H° is generated by G and its centralizer.