On the logical structure of physics

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A model theory ussue

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Claim. Post-Newtonian physics speaks in the language of **continuous logic**

1. Probabilistic calculus in Boltzmann's statistical mechanics

Probability of a state with energy E at temperature T is

$$\frac{E}{kT} \in \mathbb{U} \mapsto \frac{1}{Z_T} \mathrm{e}^{\frac{E}{kT}} \in \mathbb{R}; \qquad \mathbb{U} \to \mathbb{R}$$

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2. Probabily amplitudes calculus in Dirac's quantum mechanics

Probability amplitude of a state with energy H at time t is

$$\frac{Ht}{\hbar} \in \mathbb{U} \mapsto \frac{1}{c_t} \mathrm{e}^{\frac{iHt}{\hbar}} \in \mathbb{C}; \qquad \mathbb{U} \to \mathbb{C}$$

Continuous predicates and quantifiers

States = n-ary CL-**predicates**

$$\psi : \mathbb{U}^n \to \mathrm{F}$$

Example.

$$\langle x \mid p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \mathrm{e}^{i\frac{px}{\hbar}}$$

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Quantifiers (bounded) ($a, p \in \mathbb{R}_+$)

$$\mathrm{e}^{px} \mapsto \int_{\mathbb{R}} \mathrm{e}^{-a\frac{x^2}{2}} \mathrm{e}^{px} dx \ = \sqrt{\frac{\pi}{a}} \mathrm{e}^{\frac{p^2}{4a}} \quad \text{(SM calculus)}$$

and

$$\mathrm{e}^{ipx} \mapsto \int_{\mathbb{R}} \mathrm{e}^{-ai\frac{x^2}{2}} \mathrm{e}^{ipx} dx \ = \sqrt{\frac{\pi}{ai}} \mathrm{e}^{-i\frac{p^2}{4a}} \quad \text{(QM calculus)}$$

Magic rules

- **Wick rotation:**
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-**Regularisations:**
E.g.: for some infinite matrix A,

$$\det A \text{ ``=''} \prod_{n \in \mathbb{N}} n \text{ ``=''} e^{\zeta'(0)} = \sqrt{2\pi}$$

...

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Problem. Give an *interpretation* of the CL-axioms in the context of continuous model theory.

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A locally finite model of gravity, 2008; *Relating the Quantum Mechanics of Discrete Systems to Standard Canonical Quantum Mechanics* 2014, ...
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Assuming the "universe" $\mathbb{U}$ is discrete or finite, the Logic System must be multivalued.
With values in some **finite or pseudo-finite** F :

$$\exp_{\mathfrak{p}} : \mathbb{U}^n \to \mathrm{F}$$

Roger Penrose in *The Road to reality* on using finite fields in physics:

... If $\mathbb{F}_p$ were to take the place of the real-number system, in any significant sense, then p has to be very large indeed. ... To my mind, a physical theory which depends fundamentally upon some absurdly enormous prime number would be a far more complicated (and improbable) theory than one that is able to depend on a simple notion of infinity. Nevertheless, it is of some interest to pursue these matters. ...

Finite and pseudo-finite models

$\mathbb{U}$ is pseudo-finite $\Rightarrow \mathrm{F}$ is pseudo-finite

$$\mathbb{U} := {}^*\mathbb{Z}/(\mathfrak{p}-1)\mathfrak{l}; \quad \mathrm{F} = {}^*\mathbb{Z}/\mathfrak{p} = \mathrm{F}_{\mathfrak{p}}$$

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$$\exp_\mathfrak{p} : \mathbb{U} \twoheadrightarrow \mathrm{F}_\mathfrak{p}^\times$$

a surjective homomorphism of groups.

Limits of finite structures

Theorem *For some non-standard prime $\mathfrak{p}$, highly divisible $\mathfrak{l} \in {}^*\mathbb{Z}$ and a very large $\mathbf{i} \in {}^*\mathbb{Z}$, there is a pair of surjective "limit" homomorphisms which make the diagram commute*

$$\begin{array}{cccc} \mathrm{lm}_{\mathbb{U}}: & \mathbb{U} & \twoheadrightarrow & \bar{\mathbb{C}} \\ & \exp_{\mathfrak{p}} \downarrow & & \exp \downarrow \ ; \\ \mathrm{lm}_{\mathrm{F}}: & \mathrm{F}_{\mathfrak{p}} & \twoheadrightarrow & \bar{\mathbb{C}} \end{array}$$

where $\mathrm{lm}_{\mathbb{U}}$ is a homomorphism of a $\mathbb{Z}[\mathbf{i}]$-modules such that

$$\mathrm{lm}_{\mathbb{U}}(\mathbf{i} \cdot \mathbf{u}) = i \cdot \mathrm{lm}_{U}(\mathbf{u}), \quad i = \sqrt{-1}$$

Limits of finite structures

There are subgroups: $'\mathbb{R}' \subset \mathbb{U}$; $\mathbf{i} \cdot '\mathbb{R}' \subset \mathbb{U}$ such that

$$\mathrm{lm}_{\mathbb{U}} : '\mathbb{R}' \twoheadrightarrow \mathbb{R} \subset \mathbb{C}$$

$$\mathrm{lm}_{\mathbb{U}} : \mathbf{i} \cdot '\mathbb{R}' \twoheadrightarrow i \cdot \mathbb{R} \subset \mathbb{C}$$

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$$\mathrm{lm}_{\mathrm{F}} : '\mathbb{R}'_{+} \twoheadrightarrow \mathbb{R}_{+} \subset \mathbb{C}$$

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This allows polar coordinates and “complex” conjugation on a “dense” subfield $\mathrm{F} \subset \mathrm{F}_{\mathfrak{p}}$, with an embedding

$$\mathrm{F} \hookrightarrow {}^{*}\mathbb{C}$$

so allows *non-standard analysis on* F.

Predicates and states with values in $\mathrm{F} \subseteq \mathrm{F}_{\mathfrak{p}}$

A state (= predicate) on $\mathbb{V} \subset \mathbb{U}^n$ is:

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A basic state (basic predicate) φ has the form

$$\varphi(\bar{x}) = \exp_{\mathfrak{p}}(f(\bar{x}) \cdot \mathbf{v})$$

where $f(\bar{x}) \in \mathbb{Z}[\bar{x}]$, $\bar{x} \in ({}^*\mathbb{Z}/\mathcal{N})^n$ and $\mathbf{v} \in \mathbb{V}$, $\mathcal{N} = |\mathbb{V}|$.
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Logical connectives = operations of F. For quantifiers use

$$\frac{1}{\sqrt{\mathcal{N}}} \sum_{y \in {}^*\mathbb{Z}/\mathcal{N}} \exp_{\mathfrak{p}}(g(\bar{x}, y) \cdot \mathbf{v})$$

(cf. E.Kowalski's and E.Hrushovski's works on additive character over finite fields).

Hilbert space over F and operators

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Definable linear maps on $\mathbb{H}_{\mathbb{V}}$, analogues of **linear unitary operators**:

$$\mathrm{L}_g : \varphi(\bar{x}, y)) \mapsto \frac{1}{\sqrt{|\mathbb{V}|}} \sum_{n\mathbf{v} \in \mathbb{V}} \exp_{\mathfrak{p}}(g(n)\mathbf{v}) \cdot \varphi(\bar{x}, n)$$

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which interpret the continuous logic quantifiers

$$\mathrm{L}_g : \varphi(\bar{x}, y)) \mapsto \int_{\mathbb{R}} \mathrm{e}^{g(y)} \varphi(\bar{x}, y)) \, dy$$

or

$$\mathrm{L}_g : \varphi(\bar{x}, y)) \mapsto \int_{\mathbb{R}} \mathrm{e}^{ig(y)} \varphi(\bar{x}, y)) \, dy$$

Wick rotation on Gaussian states

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Theorem. For some natural subgroup-subdomains

$$\mathbb{V}_{\mathrm{QM}} \subset \mathbb{V}_{\mathrm{SM}} \subset \mathbb{U}^n$$

$$\mathbb{V}_{\mathrm{QM}} = \mathbf{i} \cdot \mathbb{V}_{\mathrm{SM}};$$

$$\mathbf{u} \mapsto \mathbf{i} \cdot \mathbf{u} \text{ induces } \varphi \mapsto \varphi^{\mathbf{i}}$$

where

$$\varphi : \mathbb{V}_{\mathrm{SM}} \to \mathrm{F}; \quad \varphi^{\mathbf{i}} : \mathbb{V}_{\mathrm{QM}} \to \mathrm{F}$$

And for linear Gaussian operators L on $\varphi \in \mathbb{H}_{\mathrm{SM}}$ becomes the action of some well-defined linear operator $\mathsf{L}^{\mathbf{i}}$ on $\varphi^{\mathbf{i}} \in \mathbb{H}_{\mathrm{QM}}$,

$$\mathsf{L}^{\mathbf{i}}\varphi^{\mathbf{i}} = (\mathsf{L}\varphi)^{\mathbf{i}}.$$

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The inner product on the spaces transforms correspondingly

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This results in the **Wick rotation isomorphism**

$$\{\}^{\mathbf{i}} : \ \mathbb{H}_{\mathrm{SM}} \to \mathbb{H}_{\mathrm{QM}}$$