Morley Rank, non-forking, canonical bases

Seminar on ample theories

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Source This talk contains material from classical textbooks, mostly from chapters 6,7 and 8 of Tent-Ziegler and chapter 6 and 8 from Marker.

Conventions We write lowercase letters for single elements or finite tuples. We write "*subset*" for subsets – proper or not. We write "$\triangle$" for symmetric difference of formula: $\varphi\triangle\psi = (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$. We write "$\sqcup$" for disjoint unions of formulas, so $\varphi \sqcup \psi$ is the formula $\varphi \vee \psi$, but it is only defined on disjoint formulas. We write *we* for *many cups of coffee, the induced lack of sleep and I.*

Morley Rank

Ordinal-valued notion of dimension for formulas or definable sets.

Definition and basic properties

In a given structure $\mathcal{M}$, we define by induction:

- $\mathrm{MR}_{\mathcal{M}}(\varphi) \geqslant 0$ iff φ is consistent;
- $\mathrm{MR}_{\mathcal{M}}(\varphi) \geqslant \alpha + 1$ iff there are $(\varphi_i)_{i<\omega}$ disjoint, each implying φ, and each of MR $\geqslant \alpha$;
- $\mathrm{MR}_{\mathcal{M}}(\varphi) \geqslant \lambda$ iff $\mathrm{MR}(\varphi) \leqslant \alpha$ for all $\alpha < \lambda$.

Note that φ and φ_i are allowed to have different parameters.

Given a complete theory T, we define $\mathrm{MR}_T(\varphi(x, a))$ to be $\mathrm{MR}_{\mathcal{M}}(\varphi)$ for $\mathcal{M} \vDash T$ $\aleph_0$-saturated and containing a.

Lemma 1. $\mathrm{MR}_T(\varphi)$ *is well defined, ie, it does not depend on the choice of an $\aleph_0$-saturated model.*

Proof.

- If $\mathcal{M}$ is $\aleph_0$-saturated and $\mathrm{tp}_{\mathcal{M}}(a) = \mathrm{tp}_{\mathcal{M}}(b)$, then $\mathrm{MR}_{\mathcal{M}}(\varphi(x, a)) = \mathrm{MR}_{\mathcal{M}}(\varphi(x, b))$.
  - If MR=0 it's clear, we proceed by induction.
  - Anytime $\varphi(x, a) = \varphi_1(x, c_1) \sqcup \cdots \sqcup \varphi_n(x, c_n)$, by saturation we can find $d_1, \cdots, d_n$ such that $\mathrm{tp}_{\mathcal{M}}(a, \overline{c}) = \mathrm{tp}_{\mathcal{M}}(b, \overline{d})$.
- If $\mathcal{M} \preccurlyeq \mathcal{N}$ are both $\aleph_0$-saturated, then $\mathrm{MR}_{\mathcal{M}}(\varphi) = \mathrm{MR}_{\mathcal{N}}(\varphi)$.
  - If MR=0 it's clear, we proceed by induction.
  - $\mathrm{MR}_{\mathcal{M}}(\varphi) \leqslant \mathrm{MR}_{\mathcal{N}}(\varphi)$ is clear.
  - In the other direction, we might have parameters from $\mathcal{N}$, but we can replace them by parameters from $\mathcal{M}$ by saturation.
- Let $\mathcal{M} \vDash T$ containing parameters of φ. Then: $\begin{array}{ccccc} & & \mathcal{N}_2 & & \\ & \preccurlyeq & & \succcurlyeq & \\ \mathcal{M} & & & & \mathcal{N}^* \\ & \succcurlyeq & & \preccurlyeq & \\ & & \mathcal{N}_1 & & \end{array}$

□

In the following, we assume that structures are at least $\aleph_0$-saturated – equivalentely, we work in a monster.

It is easy to check that $\mathrm{MR}(\varphi \vee \psi) = \max(\mathrm{MR}(\varphi), \mathrm{MR}(\psi))$. By definition, we have $\mathrm{MR}(\varphi) = 0$ iff $\varphi(M)$ is finite (and non-empty). If φ is inconsistent, we write $\mathrm{MR}(\varphi) = -\infty$. Since the Morley rank of $\varphi(x, a)$ only depends on $\mathrm{tp}(a)$ and there is no gap in the values of MR, we have:

Proposition 2. *If* $\mathrm{MR}(\varphi) > (2^{|T|})^+$*, then* $\mathrm{MR}(\varphi) \geqslant \alpha$ *for all* α*; we write* $\mathrm{MR}(\varphi) = \infty$.

Definition 3. We define an equivalence relation $\varphi \sim_\alpha \psi$ by $\mathrm{MR}(\varphi \triangle \psi) < \alpha$.

A formula of MR α is α-strongly-minimal if for any ψ, either $\varphi \wedge \psi$ or $\varphi \wedge \neg\psi$ is of MR $< \alpha$.

If a formula is of MR α, the maximal amount of definable subsets of $\varphi(M)$ of MR α is called the Morley Degree of φ.

Proposition 4. $\mathrm{MD}(\varphi)$ *is well-defined (if* $\mathrm{MR}(\varphi)$ *is not* $\pm\infty$*): there is a decomposition* $\varphi = \varphi_1 \sqcup \cdots \sqcup \varphi_d$*, with all* φ_i *α-strongly-minimal, unique up to* $\sim_\alpha$.

Proof. It's clear that if such a decomposition doesn't exist, φ has MR $> \alpha$. To prove uniqueness, take ψ α-strongly-minimal implying φ; then there is exactly one φ_i such that $\varphi_i \wedge \psi$ is of rank α, and thus $\psi \sim_\alpha \varphi_i$. □

- α-strongly-minimal $\Leftrightarrow \mathrm{MR} = \alpha, \mathrm{MD} = 1$
- $\mathrm{MR}(\varphi) = 0 \Rightarrow \mathrm{MD}(\varphi) = |\varphi(M)|$
- 0-strongly-minimal $\Leftrightarrow |\varphi(M)| = 1$
- 1-strongly-minimal $\Leftrightarrow$ strongly-minimal as usual

Definition 5. For a type p, we define:

$$\mathrm{MR}(p) = \min_{\varphi \in p}(\mathrm{MR}(\varphi)), \text{ and } \mathrm{MD}(p) = \min_{\varphi \in p, \mathrm{MR}(\varphi) = \mathrm{MR}(p)}(\mathrm{MD}(\varphi))$$

We also write $\mathrm{MR}(A/B) = \mathrm{MR}(\mathrm{tp}(A/B))$ and similarly for MD.

We have $\mathrm{MR}(A/B) = 0$ iff $A \subset \mathrm{acl}(B)$.

In strongly minimal theories

Recall that acl is a pregeometry on strongly-minimal theories:

- $A \subset B \Rightarrow \mathrm{acl}(A) \subset \mathrm{acl}(B)$
- $A \subset \mathrm{acl}(A)$
- $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$
- $\mathrm{acl}(A) = \bigcup_{A_0 \subset A \text{ finite}} \mathrm{acl}(A_0)$
- $b\, \mathrm{acl}(Ac) \setminus \mathrm{acl}(A) \Rightarrow c \in \mathrm{acl}(Ab)$

The last property, called Exchange, might fail outside of strongly minimal theories; the four others always hold.

Recall that a basis for A over B is a subset A' such that $\mathrm{acl}(A'B) = \mathrm{acl}(AB)$ and for any $X \subsetneq A'$, $\mathrm{acl}(XB) \subsetneq \mathrm{acl}(AB)$. $\dim(A/B)$ is the cardinal of a basis of A over B; this is well defined.

We write $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if $\dim(a/C) = \dim(a/BC)$ for all finite $a \in A$.

Theorem 6. *In a strongly minimal theory, we have:*[*]

$$\mathrm{MR}(a_1, \cdots, a_n/B) = \dim(a_1, \cdots, a_n/B)$$

[*]here a_1 is a point, not a tuple.

For $n = 1$, either $\dim(a/B) = 0 \Leftrightarrow a \in \operatorname{acl}(B) \Leftrightarrow \mathrm{MR}(a/B) = 0$, or $\dim(a/B) = 1 \Leftrightarrow a \notin \operatorname{acl}(B) \Leftrightarrow \mathrm{MR}(a/B) \geqslant 1$. By strong minimality, any formula with 1 free variable is of MR$\leqslant 1$, so we are done.

The strategy for arbitrary n is the same: first we deal with the case $\dim < n$, then with the case $\dim = n$.

Lemma 7. *If $b \in \operatorname{acl}(Ca)$, $\mathrm{MR}(b/C) \leqslant \mathrm{MR}(a/C)$.*

Proof. We work by induction on $\alpha = \mathrm{MR}(a/C)$. If $\alpha = 0$, it is clear that $\mathrm{MR}(b/C) = 0$.

We have $\mathrm{MR}(b/Ca) = 0$, let $d = \mathrm{MD}(b/Ca)$. We take $\psi_1(x) \in \operatorname{tp}(a/C)$ of MR α, and we take $\psi_2(a, y) \in \operatorname{tp}(b/Ca)$ of MR 0 and MD d. We may assume $\mathcal{M} \vDash \forall x \exists^{\leqslant d} y\ \psi_2(x, y)$. Now let $\varphi(x, y) = \psi_1(x) \wedge \psi_2(x, y)$. We have:

$$\mathrm{MR}(\exists y \varphi(x, y)) = \alpha \text{ and } |\varphi(a', M)| \leqslant d.$$

Consider $\chi(y) = \exists x \varphi(x, y)$. We will prove $\mathrm{MR}(\chi) \leqslant \alpha$; since $\chi \in \operatorname{tp}(b/A)$, this proves $\mathrm{MR}(b/A) \leqslant \alpha$.

Let χ_i be an infinite family defining disjoint subsets of χ, say with parameters in C'. Let $\psi_i(x) = \exists x(\varphi(x, y), \chi(y))$. ψ_i implies $\exists y \varphi(x, y)$, and since any $d + 1$ of the ψ_i are disjoint, at least one of them must have MR $< \alpha$.

Take any b' realizing $\chi_i(y)$. Then by definition of χ there is a' realizing $\varphi(a', b')$. So $b' \in \operatorname{acl}(C'a')$ and $\mathrm{MR}(a'/C') \leqslant \mathrm{MR}(\psi_i) < \alpha$, so by induction, $\mathrm{MR}(b'/C') < \alpha$. Because this is true for any b', we conclude $\mathrm{MR}(\chi_i) < \alpha$. □

This lemma allows us to only consider the case where the a_i are independent over B, that is, $\dim(a_1, \cdots, a_n/B) = n$.

Proposition 8. *In a strong minimal theory, the type of n independent elements over a given subset is uniquely determined.*

Proof. In dim 1 it's clear and has been done last week, prove the rest by induction. □

Thus any formula on n free variables must have MR$\leqslant n$, so in particular, $\mathrm{MR}(a_1, \cdots, a_n/B) \leqslant n$.

Remains to prove that if $\dim(a_1, \cdots, a_n/B) = n$, $\mathrm{MR}(a_1, \cdots, a_n/B) = n$.

- $\mathrm{MR}(a_1, \cdots, a_n/Ba_1) = n - 1$.
- Let $\psi \in \operatorname{tp}(a_1, \cdots, a_n/B)$. $\chi(\overline{x}, a_1) = \psi(x_1, \cdots, x_n) \wedge x_1 = a_1$ has MR$\geqslant n - 1$.
- If $\operatorname{tp}(a/B) = \operatorname{tp}(a_1/B)$, $\mathrm{MR}(\chi(\overline{x}, a)) = n - 1$.
- $\{\chi(\overline{x}, a) \mid a \equiv_B a_1\}$ is a disjoint family of subsets of ψ, so $\mathrm{MR}(\psi) \geqslant n$.

$\square_{thm}$

Wanna fork?

From now on T is ω-stable; this is equivalent (in a countable language) to saying that the Morley Rank is never ∞.

Definition 9.

- We write $A \mathop{\smile\!\!\!\!\!|}_C B$ when $\mathrm{MR}(a/C) = \mathrm{MR}(a/BC)$ for any finite $a \in A$.
- We say that $\mathrm{tp}(a/BC)$ forks over C when $a \not\mathop{\smile\!\!\!\!\!|}_C B$.
- For $A \subset B$, $p \in S_n(A)$, $q \in S_n(B)$, we say that the extension $p \subset q$ is forking if $\mathrm{MR}(p) > \mathrm{MR}(q)$, or equivalently if q forks over A.
- $p \in S_n(C)$ is called stationary if for any $C \subset D$, p has a unique non-forking extension to D.

One can define forking in arbitrary theories but who has time for that? Certainly not us.

Lemma 10. *If $p \in S_n(A)$ has MD d and $A \subset \mathcal{M}$, then there are exactly d non-forking extension of p in $S_n(\mathcal{M})$, and they are of MD 1.*

Proof. If $\varphi \in p$ realizes MR and MD of φ and $\varphi = \varphi_1 \sqcup \cdots \sqcup \varphi_d$, then complete types over $\mathcal{M}$ containing $p \cup \{\varphi_i\}$ are exactly the non-forking extensions of p. □

Proposition 11. *If $\mathcal{M}$ is κ-saturated and κ-homogeneous, any type forking over a subset A smaller than κ has at least κ many conjugates over A.*

No proof given.

Theorem 12 (Characterization of non-forking)**.** *T is stable if and only if there is a special class of extensions of n-types, which we denote by $p \sqsubset q$, with the following properties:*

1. *(Invariance) $\sqsubset$ is invariant under* $\mathrm{Aut}(\mathcal{M})$,
2. *(Local character) There is a cardinal κ such that for $q \in S_n(M)$ there is $C_0 \subset C$ of cardinality at most κ such that $q|C_0 \sqsubset q$.*
3. *(Weak Boundedness) For all $p \in S_n(A)$ there is a cardinal μ such that p has, for any $A \subset B$, at most μ extensions $q \in S_n(B)$ with $p \sqsubset q$.*

If $\sqsubset$ satisfies in addition:

4. *(Existence) For all* $p \in S_n(A)$ *and* $A \subset B$*, there is* $q \in S_n(B)$ *such that* $p \sqsubset q$*,*

5. *(Transitivity)* $p \sqsubset q \sqsubset r$ *implies* $p \sqsubset r$*,*

6. *(Weak Monotonicity)* $p \sqsubset r$ *and* $p \subset q \subset r$ *implies* $p \sqsubset q$*,*

then $\sqsubset$ *coincides with the non-forking relation.*

If we have time, we will prove that in stable theories, those conditions characterize non-forking.

Canonical bases

Definition 13.

- $a \in \mathcal{M}$ is called a canonical parameter for a definable set $D \subset \mathcal{M}$ if for any $\sigma \in \mathrm{Aut}(\mathcal{M})$, $\sigma(a) = a$ iff D is invariant under σ.

- $A \in \mathcal{M}$ is called a canonical base for a type p if any $\sigma \in \mathrm{Aut}(\mathcal{M})$ fixes A pointwise iff p is invariant under σ.

Lemma 14. *Any definable set has an imaginary canonical parameter, that is, a canonical parameter in* $\mathcal{M}^{eq}$*.*

Proof. Write $X = \varphi(M, a)$. Define $x \sim y$ by $\varphi(M, x) = \varphi(M, y)$. $(a/\sim) \in \mathcal{M}^{eq}$ is a canonical parameter for X. □

Note that the canonical parameter lies in $\mathrm{dcl}^{eq}(a)$, also, EI is equivalent to saying each set has a (real) canonical parameter.

Lemma 15. *Any definable type has an imaginary canonical base.*

Proof.

- $B_\varphi = \{b \in \mathcal{M} \mid \varphi(x, b) \in p\}$ is definable by assumption

- $\sigma p = p$ iff $\sigma(B_\varphi) = B_\varphi$ for all φ

- Since B_φ is definable, it has a canonical parameter $a_\varphi \in \mathcal{M}^{eq}$

- $A = \{a_\varphi \mid \varphi \in \mathcal{L}\}$ is an imaginary canonical base for p.

□

Proposition 16. *In an* ω*-stable theory, for any formulas* $\varphi(x, a)$ *of MR* α *and* $\psi(x, y)$*, the set* $\{b \in \mathcal{M} \mid \mathrm{MR}(\varphi(x, a) \wedge \psi(x, b)) = \alpha\}$ *is* a*-definable.*

Proof. Write $\chi(x,b)$ for $\varphi(x,a) \wedge \psi(x,b)$.

- We might assume $\mathrm{MD}(\varphi) = 1$.
- If $\mathrm{MR}(\chi(x,c)) = \alpha$, there is a finite $X_c \subset \chi(\mathcal{M},c)$ such that if $X_c \subset \psi(\mathcal{M},b)$, $\mathrm{MR}(\chi(x,b)) = \alpha$.
  - Chose $a_0 \in \chi(\mathcal{M},c)$. If we can't find b_0 such that $a_0 \in \psi(\mathcal{M},b_0)$ and $\mathrm{MR}(\chi(x,b_0)) < \alpha$, then we can take $X_c = a_0$. Otherwise we take such a b_0 and continue by induction.
  - We have $\mathrm{MR}(\chi(x,c) \wedge \bigwedge_{i \leqslant n} \neg\psi(x,b_i)) = \alpha$, so we can take a_{n+1} in there. Once again, if $X_c = \{a_0, \cdots, a_n\}$ works, we're done; otherwise there is b_{n+1} such that $\{a_0, \cdots, a_{n+1}\} \subset \psi(x,b_{n+1})$ and $\mathrm{MR}(\chi(x,b_{n+1})) < \alpha$.
  - It has to stop because $\psi(a_i,b_j)$ holds iff $i \leqslant j$.
- Let $Y = \{X \subset \varphi(\mathcal{M},a) \text{ finite} \mid X \subset \psi(\mathcal{M},b) \Rightarrow \mathrm{MR}(\chi(x,b)) = \alpha\}$ and let $\theta_X(y) = \bigwedge_{x \in X} \psi(x,y)$. Now $\mathrm{MR}(\chi(x,b)) = \alpha$ iff $\bigvee_{X \in Y} \theta_X(b)$.
- We have the same result for $\neg\psi$.
- We move to a pair $(\mathcal{M},\mathcal{M}^*)$ where $\mathcal{M} \preccurlyeq M^*$ and $\mathcal{M}^*$ is $\mathcal{M}$-saturated. We consider φ on $\mathcal{M}$ and ψ on $\mathcal{M}^*$.
- By saturation we can finitize the disjunction, thus we have definability.
- If $\sigma(a) = a$, $\mathrm{MR}(\chi(x,b)) = \mathrm{MR}(\chi(x,\sigma(b)))$; thus we have a-definability.

$\square$

Let $\varphi \in p$ realize MR and MD of p, then $p = \{\psi \mid \mathrm{MR}(\varphi \wedge \psi) = \alpha\}$, thus:

Corollary 17. *In an ω-stable theory, any $p \in S_n(A)$ is definable over a finite $A_0 \subset A$, and thus has a finite imaginary canonical base in $\mathrm{dcl}^{eq}(A_0)$.*

Theorem 18. *In an ω-stable theory, if $p \in S_n(A)$ and $\mathrm{MD}(p) = 1$, then any non-forking extension q of p is A-definable.*

If $\mathrm{MD}(p) > 1$, there is $a \in \mathrm{acl}^{eq}(A)$ such that q is Aa-definable.

Proof. In degree 1, we have $q = \{\psi(x,b) \mid \mathrm{MR}(\varphi \wedge \psi) = \alpha, b \in B\} \in S_n(B)$, so we might use the same definition than for p, needing only parameters appearing in φ.

In degree $d > 1$:

- Fix $\mathcal{M}$ containing B, q (hence p) has a non-forking extension q^* to $\mathcal{M}$. q^* is of MD 1.

- Take $\varphi(x) \in p$ and $\psi(x,b) \in q^*$ realizing MR and MD of p and q^*. We can assume ψ implies φ.

- $X = \{c \mid \mathrm{MR}(\psi(x,c)) = \alpha$ and $\forall d$, if $\mathrm{MR}(\psi(x,d)) = \alpha$, then either $\mathrm{MR}(\psi(x,c) \wedge \psi(x,d)) < \alpha$ or $\mathrm{MR}(\psi(x,c) \wedge \neg\psi(x,d)) < \alpha\}$ is A-definable.

- $c \sim c'$: $\mathrm{MR}(\psi(x,c) \wedge \psi(x,c')) = \alpha$ is a definable equivalence relation on X.

- $|X/\sim| \leqslant d$.

- For any $c \sim b$, $q = \{\chi \mid \mathrm{MR}(\chi \wedge \psi(x,c) = \alpha)\}$, and q is Ac-definable.

- Because $\sim$ has finitely many classes and is A-def, $a = (b/\sim) \in \mathrm{acl}^{eq}(A)$, and q is Aa-def.

□

Corollary 19. *In an ω-stable theory, if $p \in S_n(\mathcal{M})$ doesn't fork over A, then p has a canonical base in $\mathrm{acl}^{eq}(A)$. If $p|A$ is stationary, p has a canonical base in $\mathrm{dcl}^{eq}(A)$.*

Proof. Fix $\varphi(x,a)$ realizing $\mathrm{MR}(p)$, let $X = \{b \mid \varphi(x,b) \in p\}$. X is $\mathrm{acl}(A)^{eq}$ (resp. A)- definable and $\sigma p = p$ iff $\sigma(X) = X$. Now X has a canonical parameter in $\mathrm{dcl}^{eq}(\mathrm{acl}^{eq}(A))$ (resp. $\mathrm{dcl}^{eq}(A)$). □

If p has a canonical base A, we write $\mathrm{cb}(p) = \mathrm{dcl}^{eq}(A)$. This is well-defined.

Corollary 20. *In an ω-stable theory, $p \in S_n(\mathcal{M})$ doesn't fork over A iff $\mathrm{cb}(p) \subset \mathrm{acl}^{eq}(A)$.*

Reformulating, $a \mathop{\smile\!\!\!\!\!\mid}_C B$ iff $p = tp(a/BC)$ doesn't fork over C iff $\mathrm{cb}(p) \subset \mathrm{acl}^{eq}(C)$.

Corollary 21. *If $A = \mathrm{acl}^{eq}(A)$, $p \in S_n(A)$ is stationary.*

Proof. Let φ realise MR and MD of p. If MD=1, we're done. If not, I can write $\varphi(x) = \varphi_1(x,b_1) \sqcup \cdots \sqcup \varphi_d(x,b_d)$. Let $q_i \in S_n(\mathcal{M})$ be the non-forking extension of p containing φ_i. $\mathrm{cb}(q_i) \subseteq \mathrm{acl}^{eq}(A) = A$, so we might assume $b_i \in \mathrm{acl}^{eq}(A)$. But then $\varphi(x,b_i)$ or $\neg\varphi(x,b_i)$ must be in p already; so we must have $d = 1$. □

Proving non-forking characterization

Let $\sqsubset$ check the conditions of theorem 12 and take $p \in S_n(A)$, $q \in S_n(B)$, $p \subset q$. If $p \sqsubset q$:

- By 3 (Boundedness) there is μ such that p has at most μ $\sqsubset$-extensions to $S_n(B)$.
- By proposition 11 we can take $B \subset \mathcal{M}$ such that any $r \in S_n(\mathcal{M})$ forking over A has at least μ conjugates over A.
- By 4 (Existence) and 5 (Transitivity) we can find $p \sqsubset r \in S_n(\mathcal{M})$.
- By 1 (Invariance) $p \sqsubset r'$ for any conjugate; since there can only be $< \mu$, they are non-forking.

The other direction needs the full power or canonical bases.

Lemma 22. *In an ω-stable theory, for $p \in S_n(A)$ and $\kappa > max(|T|, |A|)$, in any $\mathcal{M}$ strongly κ-homogeneous, all non-forking extensions of p to $\mathcal{M}$ are conjugate.*

Proof. Let q_1, q_2 be extensions of p.

- In $\mathcal{M}^{eq}$, there is an A-automorphism of $\mathrm{acl}^{eq}(A)$ sending $q_1|\,\mathrm{acl}^{eq}(A)$ to $q_2|\,\mathrm{acl}^{eq}(A)$.
- By strong homogeneity, the reduct of this to the base sort extends to an A-automorphism σ of $\mathcal{M}$, which in turns corresponds to an A-automorphism σ^{eq} of $\mathcal{M}^{eq}$.
- Now $\sigma^{eq} q_1$ is a non-forking extension of $q_2|\,\mathrm{acl}^{eq}(A)$; but by stationarity this must be q_2.

□

Now we take $p \subset q$ non-forking:

- Take $\mathcal{M}$ strongly κ-homogeneous for a large enough κ, take $q \subset r \in S_n(\mathcal{M})$ non-forking and $p \sqsubset r' \in S_n(\mathcal{M})$.
- We know $p \subset r'$ is non-forking.
- By the previous lemma r and r' are conjugate, so $p \sqsubset r$.
- By 6 (Monotonicity) we have $p \sqsubset q$.