The free pseudospaces.

§ Set-up

Let $\mathcal{L}_n$ be the language $L_n := \{V_0, \dots, V_n, E\}$ where $V_0, \dots, V_n$ are predicates and E a binary relation.

Def:

1. An Ln-graph is an n+1-colored graph with vertices $\bigsqcup_{0 \le i \le n} V_i$, and

   $E \subseteq \bigcup_{1 \le i \le n} (V_{i-1} \times V_i) \cup (V_i \times V_{i-1})$ (E is symmetric).

   We call V_i the level i.

2. Two vertices $x_i \in V_i$, $x_j \in V_j$ are called incident if there are $x_\ell \in V_\ell$ for ~~ℓ∈~~ $\min\{i,j\} \le \ell \le \max\{i,j\}$, s.t. $E(x_{\ell-1}, x_\ell)$ $\forall \ell > \min\{i,j\}$.

   We call such a sequence $(x_i, x_{i+1}, \dots, x_j)$ a dense flag. (x is incident with itself).

   Note the ~~orde~~ levels of x_ℓ in a dense flag can be increasing or decreasing.

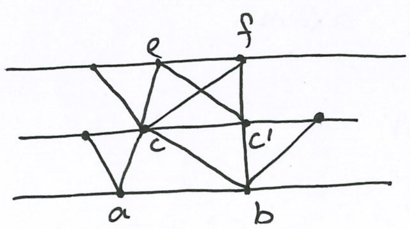
   A flag is a (not necessarily consecutative) subsequence of a dense flag.

3. The residue $R(x)$ of a vertex x is the set of vertices ~~indic~~ incident with x. We write $R_<(x)$ as the set of elements in $R(x)$ ~~that~~ of levels less than the level of x. And $R_\le(x) := R_<(x) \cup \{x\}$. Similarly we have $R_>(x)$, $R_\ge(x)$.

4. Say x, y intersect in z (or intersect in the empty set), write as $z = x \wedge y$ (or $\phi = x \wedge y$) if $R_\le(x) \cap R_\le(y) = R_\le(z)$ (or $R_\le(x) \cap R_\le(y) = \phi$)

   x, y generate z (or generate the empty set), write as $z = x \vee y$ ($\phi = x \vee y$) if $R_\ge(x) \cap R_\ge(y) = R_\ge(z)$ (or $R_\ge(x) \cap R_\ge(y) = \phi$).

5. A simple cycle is a cycle without repetitions.



$c = a \vee b$

$c \ne e \wedge f$ ($e \wedge f$ is not ~~ed.~~ defined).

§ Free pseudo-space of dimension n.

Def: A free pseudospace of dimension 0 is an infinite set of vertices. Assume that a free pseudospace of dimension $n-1$ has been defined. Then a free pseudospace of dimension n is an L_n-graph s.t.

$(\Sigma_1)_n$ (a) $(V_0, \cdots, V_{n-1}, E\restriction_{V_0 \cup \cdots \cup V_{n-1}})$ is a free pseudospace of dimension $n-1$.

(b) $(V_1, \cdots, V_n, E\restriction_{V_1 \cup \cdots \cup V_n})$ is a free pseudospace of dim. $n-1$.

$(\Sigma_2)_n$ (a) For all $x \in V_0$, $R_>(x)$ is a free pseudospace of dim $n-1$.

(b) For all $x \in V_n$, $R_<(x)$ is a free pseudospace of dim $n-1$.

$(\Sigma_3)_n$ (a) Any two vertices x and y intersect in a vertex or the empty set.

(b) Any two vertices x and y generate a vertex or the empty set.

$(\Sigma_4)_n$ (a) If a is a vertex of type V_0 and $b, b' \in R_>(a)$ with $b' \notin R(b)$ are connected by a path γ of length k, s.t. for some dense flags $f = (a, \cdots, b)$ and $f' = (b', \cdots, a)$ the concatenation of these paths $f \circ \gamma \circ f'$ is a simple cycle, then there is a path γ' of length at most k from b to b' in $R_>(a)$ containing some interior vertex γ such that $f \circ \gamma' \circ f'$ is a simple cycle.

(b) If a is a vertex of type V_n and $b, b' \in R_<(a)$ with $b' \notin R(b)$ are connected by a path γ of length k, such that for some dense flags $f = (a, \cdots, b)$ and $f' = (b', \cdots, a)$ the concatenation $f \circ \gamma \circ f'$ ~~of~~ is a simple cycle, then there is a path γ' of length at most k from b to b' in $R_<(a)$ containing some interior vertex of γ, s.t. $f \circ \gamma' \circ f'$ is a simple cycle.

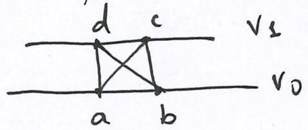
Let T_n be the L_n-theory expressing these axioms.

Thm: T_n is consistent and complete.

Remarks:

1. Induced subgraphs on $V_j \cup \cdots \cup V_{j+m}$ is a free pseudospace of dim m
2. If $a \in V_i$, then $R_<(a)$ is a free pseudospace of dim $i-1$.
   $R_>(a)$ is a free pseudospace of dim $n-i-1$
3. Self-dual: ~~if~~ define $W_i := V_{n-i}$ $\forall i = 0, \ldots, n$. Then $W_0, \ldots, W_n$ with the same set of edges is a free pseudospace of dim n.

Free pseudo space of dim 1:

Σ_3: 

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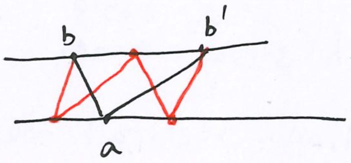
Σ_4: 

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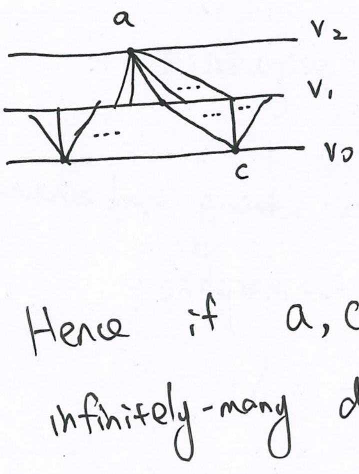
Free pseudoplane: An L_1-graph such that any vertex has infinitely many neighbours and does not contain any cycle.

Free pseudospace of dim 2:



$R_<(a)$ is a free pseudo plane. $(\Sigma_2)_2$

If $R_<(a) \cap R_>(c)$ is not empty, ~~then~~ i.e. if $c \in R_<(a)$, then $R_<(a) \cap R_>(c)$ is an infinite set, i.e. free pseudospace of dim 0 by $(\Sigma_2)_1$.

Hence if a, c are connected by a dense flag, then there are infinitely-many dense flages between a and c.

§ Consistency - Hrushovski limit.

Def: Let A be a finite L_n-graph. The following extensions are called minimal strong extensions of A.

1. Add a vertex of any type (of any level) to A which is connected to at most one vertex of A.
2. If ~~$(x_0, x_1,$~~ $(x_0, x_1, \ldots, x_k, x_{k+1})$ is a dense flag in A, add vertices $y_1, \ldots, y_k$ such that $(x_0, y_1, \ldots, y_k, x_{k+1})$ is again a dense flag.

We say B is a strong extension of A, written $A \leq B$, if B arises from A by a sequence of finitely many minimal strong extensions.

Def. Let K_n be the class of finite L_n-graphs A such that $\emptyset \leq A$.

Def. For finite L_n-graphs $A \leq B, C$, we denote $B \otimes_A C$ the free amalgam of B and C over A, i.e. the graph on $B \cup C$ containing no edges between elements of $B \setminus A$ and $C \setminus A$.

Lemma 1: If $A \leq B$, $A \leq C$ are in K_n, then $D := B \otimes_A C \in K_n$ and $B \leq D$, $C \leq D$.

Pf: It is enough to show $B \leq D$ and $C \leq D$, and it is obvious.

Def: (Strong Fraïssé Limits, Ziegler 2011).
Let K be a countable class of L-structures with a ctble class of embd. ~~strong embd~~. between elems. of K which is closed under composition. We call these embeddings strong embeddings. Assume $\emptyset \in K$ and $\emptyset \to A$ are ~~are~~ strong for all $A \in K$.

A sequence $A_0 \to A_1 \to \cdots$ of strong embeddings in K ~~are~~ is called rich, if for all i and all strong $f: A_i \to B$, $\exists j \geq i$ and a strong $g: B \to A_j$, s.t. gf is the given map $A_i \to A_j$.

A strong Fraïssé limit of K is a direct limit of a rich sequence.

Thm: If K has the amalgamation property with respect to strong embeddings, rich sequences exist and the strong Fraïssé limits are all isomorphic.

Cor: K_n has a strong Fraïssé limit, called M_n.

Thm 1: M_n is a model of T_n.

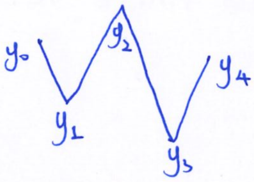
§ Completeness - Back-and-forth.

Def: 1. Let $\gamma = (a = x_0, \dots, x_m = b)$ ~~the~~ be a path, we say γ changes direction in x_i if $x_i \in V_j$ and either $x_{i-1}, x_{i+1} \in V_{j-1}$ or $x_{i-1}, x_{i+1} \in V_{j+1}$.

Rmk: A path that never changes direction is a dense flag.

2. Let $\gamma = (y_0, \dots, y_1, \dots, y_2, \dots, \dots, y_{k+1})$ be a path changing direction exactly in $y_1, \dots, y_k$. We say γ is reduced if for all $i = 0, \dots, k-1$,

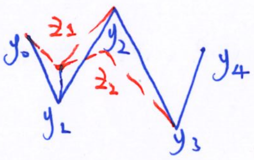
$$y_i \vee y_{i+2} = y_{i+1} \quad \text{or} \quad y_i \wedge y_{i+2} = y_{i+1}.$$



$y_0 \wedge y_2 = y_1$, $y_1 \vee y_3 = y_2$, $y_2 \vee y_4 = y_3$.

Lemma 2: If $\gamma = (a, \dots, b) \subseteq V_j \cup \dots \cup V_{j+m}$ has length s, there is a reduced path from a to ~~be~~ b inside $V_j \cup \dots \cup V_{j+m}$ of length at most s.

Pf:



$z_1 := y_0 \wedge y_2$

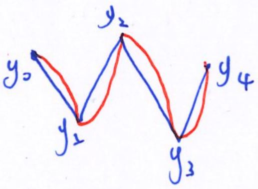
$z_2 := z_1 \vee y_3$

$\vdots$

(Note that $z_1 \in R_\leq(y_2)$, $y_3 \in R_\leq(y_2)$ $\Rightarrow z_2 \in R_\leq(y_2)$ and $z_1 = y_0 \wedge y_2 \Rightarrow z_1 = y_0 \wedge z_2$.)

Lemma 3: If $a, b \in R_\leq(C)$ are connected by a reduced path γ in $M \models T_n$, then $\gamma \subseteq R_\leq(C)$.

Def: For two (reduced) paths γ_1, γ_2 from a to b, we call γ_1, γ_2 equivalent, $\gamma_1 \sim \gamma_2$ if they have the same set of vertices in which they change direction.



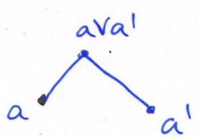
(Note that if $M \models T_n$, ~~then~~ and y_0 ~~connects~~ is connected with y_1 by a dense flag, then there are infinitely many such ~~fla~~ dense flags)

Rmk: If γ_1 is reduced and $\gamma_2 \sim \gamma_1$, then γ_2 is reduced as well.

Def: Let $M \models T_n$ and $A \subseteq M$. We call A nice in M if $A \in K_n$, i.e. A is finite and $\emptyset \leq A$, and the following holds:

1. if $a, b \in A$ are connected in M by a reduced path γ ~~of length~~ k ~~contained in $V_i \cup \dots \cup V_i$~~ in M, there is an equivalent path γ' from a to b inside A. (in particular γ' is reduced).
2. if $a, a' \in A$, then $a \vee a' \in A$ if $a \vee a'$ exists in M.
3. if $a, a' \in A$, then $a \wedge a' \in A$ if $a \wedge a'$ exists in M.

Rmk: ① 1 $\Rightarrow$ 2 and 3.



this is a reduced path in M if $a \vee a'$ exists.

~~Lemma: Let $M \models T_n$ and let $A \subseteq B$ be finite subsets of M such that B is nice in M, the A is also nice in M.~~

② If A is nice in M and $a,b \in A$. Suppose $\gamma = (a = x_0, \ldots, x_m = b)$ is a reduced path in M changing directions in $y_1, \ldots, y_k$, then $y_1, \ldots, y_k \in A$.

③ If A is nice in M, then $A \cap (V_j \cup \cdots \cup V_{j+m})$ is nice in $M \restriction (V_j \cup \cdots \cup V_{j+m}) \models T_m$. And ~~$A$~~ if $b \in A$, then $A \cap R_{\le}(b)$ ($A \cap R_{\ge}(b)$, $A \cap R_{>}(b)$, $A \cap R_{<}(b)$) are nice.

Lemma 4: Let $M \models T_n$ and let $A \le B$ be finite ~~such~~ subsets of M. Suppose $a, b \in A$ are connected by a path $\gamma \subset B$, then there is an equivalent path $\gamma' \subset A$. In particular, if B is nice, then so is A.

Pf: Sps $B = B_k \geq B_{k-1} \geq \cdots \geq B_0 = A$ and $B_i \le B_{i+1}$ are minimal strong extensions. Let $\gamma = (a = x_0, \ldots, x_n = b)$ and let j be minimal with $\gamma \subseteq B_j$. If $B_{j-1} \le B_j$ is of type 1 (add one ~~pt~~ vertex and at most one ~~vertex~~ edge), then ~~x_j is the~~ new vertex x and since the new vertex connects to at most ~~one~~ another vertex in B_j, γ contains the $x = a$ or $x = b$. ~~But~~ Hence $j = 0$ and we are done. Otherwise $B_{j-1} \le B_j$ is of type 2. So we may replace γ with an equivalent one in B_{j-1}. And we continue until get a path in A.

Key Lemma: Let $M^{\models T_n}$ be a free pseudospace. If $A \subset M$ is finite and nice in M. Let $a \in M$ be arbitrary. Then there is a nice finite set B containing $A \cup \{a\}$ such that $A \le B$.

Lemma 5: (Def: We say a model $M \models T_n$ is K_n-saturated if for all nice finite sets $A \subset M$ and strong extensions C of A with $C \in K_n$, there is a nice embedding of C into M fixing A element-wise) Let M_n be the strong Fraïssé limit of K_n, say $M_n = \bigcup_{i \in \mathbb{N}} A_i$ for the rich sequence $A_0 \le A_1 \le \cdots$. Then M_n is K_n-saturated.

Pf: If $A \le C$, ~~then~~ since $A_0 \le A_1 \le \cdots$ is a rich sequence. WMA $C =$

Thm 8: The theory T_n is complete. WMA ~~sps~~ M is ω-saturated.

Let ~~M_n~~ M be a model of T_n and let M_n be a strong limit of K_n, WMA $M_n = \bigcup_{i \in \mathbb{N}} A_i$ where $A_0 \leq A_1 \leq \cdots$ is a rich sequence.

We build a back-and-forth system: $f_0 : \emptyset \to \emptyset$.

Suppose $f_i : B_i \to B_i'$ a (partial) isomorphism with $B_i \in K_n$, ~~$B_i' = A_j$ for some j.~~ and B_i nice in M, B_i' nice in M_n.

Let $a \in M$ aribtrary, by the Key Lemma, we can find $B_{i+1} \geq B_i$ nice in M ~~contang~~ containing B_i and a. Hence $\exists\, B_{i+1}' \geq B_i'$ such that B_{i+1}' isomorphic to B_{i+1}. As $A_0 \leq \cdots$ is a rich sequence, WMA $B_i' = A_j$ and $B_{i+1}' = A_{j'}$ for some j, j'. Since any path γ in $A_{j'}$ lies in A_k for some $k = k(\gamma) \geq j'$ and $A_{j'}^0 \leq A_k$, $\gamma \in A_{j'}$. Hence $A_{j'}$ is nice in M_n. We may extend f_i to $f_{i+1} : B_{i+1} \to B_{i+1}' = A_{j'}$.

If $b \in M_n$ arbitrary. Then $b \in A_k$ for some $A_k \geq B_i'$.

By ω-saturation, there is $M \geq B_k \geq B_i$, such that $B_k \cong A_k$.

By the Key Lemma, there is ~~B_k~~ $B_{i+1} \geq B_k$ such that B_{i+1} is nice in M. Then there is $B_{i+1}' \geq A_k$ such that $B_{i+1}' = A_j \geq A_k$ and $B_{i+1}' \cong B_{i+1}$. ~~Now we~~ Similarly B_{i+1}' is nice in M_n. Now we can extend f_i to $f_{i+1} : B_{i+1} \to B_{i+1}'$ as before. $\boxtimes$

Thus, $M_n \equiv M$ and we are done.