0.1 An explicit isomorphism in Kaplan-Scanlon-Wagner

Let k be a perfect field of characteristic $p > 0$. Let ϕ be the Frobenius automorphism, $\phi(x) := x^p$, and let $\wp$ be the Artin-Schreier map, $\wp(x) := \phi(x) - x$. Let $\overline{a} = (a_0, \ldots, a_m) \in k^{m+1}$. Define $G_{\overline{a}} := \{\overline{x} | a_0\wp(x_0) = \ldots = a_m\wp(x_m)\}$. A crucial step in the Kaplan-Scanlon-Wagner proof of Artin-Schreier closedness of NIP fields is to show that if $\overline{a}$ is an algebraically independent tuple, i.e. $\operatorname{trd}(\mathbb{F}_p(\overline{a})/\mathbb{F}_p) = m + 1$, then $G_{\overline{a}}$ is isomorphic over k to the additive group, as algebraic groups. Hempel improves this by showing that the same holds when the assumption is weakened to $\mathbb{F}_p$-linear independence of $(a_0^{-1}, \ldots, a_m^{-1})$. In both cases, the proof is rather indirect, going via showing that $G_{\overline{a}}$ is connected and then referring to some standard theorems characterising vector groups in positive characteristic.

This is fine, but I thought it would be nice to actually find an isomorphism. The purpose of this note is to exhibit one. Thanks to Pierre Touchard and Mohammed Bardestani for some helpful discussion.

First we need a little lemma on a certain analogue of Vandermonde determinants. Probably it's standard, but I haven't found it in the literature. Martin Hils and Pierre Touchard remark that it is a case of a difference algebra analogue of the Wronskian.

Lemma 0.1. *Let k be a perfect field of characteristic $p > 0$. Let $\overline{c} = (c_0, \ldots, c_n) \in k^{n+1}$. Then the matrix $M := (\phi^i(c_j))_{0 \le i \le n, 0 \le j \le n}$ is singular iff $\overline{c}$ is $\mathbb{F}_p$-linearly dependent.*

Proof. It is easy to see that $\mathbb{F}_p$-linear dependence of $\overline{c}$ implies singularity of M. We prove the converse.

This is clear for $n = 0$. So suppose $n > 0$, and suppose M is singular, say $\bigwedge_{i \ge 0} \Sigma_{j \ge 0}\phi^i(c_j)\lambda_j = 0$ with $\overline{\lambda} \ne \overline{0}$, and suppose inductively the result for shorter tuples. Then the $n \times n$ matrix $(\phi^i(c_j))_{0 < i \le n, 0 < j \le n}$ is non-singular, and so since $\bigwedge_{i > 0} \Sigma_{j \ge 0}\phi^i(c_j)\lambda_j = 0 = \bigwedge_{i > 0} \Sigma_{j \ge 0}\phi^i(c_j)\phi(\lambda_j)$, we deduce that for some α we have $\bigwedge_{j \ge 0} \phi(\lambda_j) = \alpha\lambda_j$, and hence $\lambda_j = \alpha'\lambda'_j$ where $(\alpha')^{p-1} = \alpha$ and $\lambda'_j = 0$ or $(\lambda'_j)^{p-1} = 1$, i.e. $\lambda'_j \in \mathbb{F}_p$. But then $\Sigma_{j \ge 0}c_j\lambda'_j = 0$, so $\overline{c}$ is $\mathbb{F}_p$-linearly dependent. ☐

Now let $\overline{a} = (a_0, \ldots, a_m) \in k^{m+1}$, let $b_i := a_i^{-1}$, and suppose $\overline{b}$ is $\mathbb{F}_p$-linearly independent. We define an algebraic isomorphism over k of $G_{\overline{a}}$ with the additive group. So suppose $a_0\wp(x_0) = \ldots = a_m\wp(x_m)$.

Write $\delta_{i,j}$ for the Kronecker delta. By Lemma 0.1 applied to $\phi^{-m}(\overline{b})$, $(\phi^{-i}(b_j))_{i,j}$ is non-singular. Since k is perfect, $\phi^{-i}(b_j) \in k$. So there exists $\overline{\alpha} = (\alpha_0, \ldots, \alpha_m) \in k^{m+1}$ such that $\Sigma_{j \ge 0}\phi^{-i}(b_j)\alpha_j = \delta_{0,i}$, and hence $\Sigma_{j \ge 0}\frac{\phi^i(\alpha_j)}{a_j} = \delta_{0,i}$.

Claim 0.2. *$\overline{\alpha}$ is $\mathbb{F}_p$-linearly independent.*

Proof. Suppose not, so (after a permutation) we have $\alpha_0 = \Sigma_{j > 0}\lambda_j\alpha_j$ with $\lambda_j \in \mathbb{F}_p$. Then for $i > 0$, we have $\phi^{-i}(b_0)\Sigma_{j > 0}\lambda_j\alpha_j + \Sigma_{j > 0}\phi^{-i}(b_j)\alpha_j = 0$, so $\Sigma_{j > 0}(\phi^{-i}(b_j + \lambda_j b_0)\alpha_j = 0$.

But $\alpha_j \ne 0$ for some $j > 0$, since $\overline{\alpha} \ne \overline{0}$, so by Lemma 0.1, $(b_j - \lambda_j b_0)_{j > 0}$ is $\mathbb{F}_p$-linearly dependent. But then so is $\overline{b}$, contrary to assumption. ☐

Set $t := \Sigma_{i \ge 0}\alpha_i x_i$.

Claim 0.3. *For $k \geq 0$, we have $\phi^k(t) = \Sigma_{i\geq 0}\phi^k(\alpha_i)x_i$.*

Proof. This holds by definition for $k = 0$, and then inductively and using the equations of $G_{\overline{a}}$ we have

$$\begin{aligned}
\phi^{k+1}t &= \phi(\Sigma_{i\geq 0}\phi^k(\alpha_i)x_i) \\
&= \Sigma_{i\geq 0}\phi^{k+1}(\alpha_i)\phi(x_i) \\
&= \Sigma_{i\geq 0}\phi^{k+1}(\alpha_i)(\wp(x_i) + x_i) \\
&= \Sigma_{i\geq 0}\frac{\phi^{k+1}(\alpha_i)}{a_i}a_i\wp(x_i) + \Sigma_{i\geq 0}\phi^{k+1}(\alpha_i)x_i \\
&= \Sigma_{i\geq 0}\phi^{k+1}(\alpha_i)x_i.
\end{aligned}$$

□

Now by Claim 0.2 and Lemma 0.1, the matrix $(\phi^i(\alpha_j))_{i\geq 0, j\geq 0}$ is non-singular, so say $(\beta_{ij})_{i\geq 0, j\geq 0}$ is the inverse, $\beta_{ij} \in k$.

Then $x_i = \Sigma_{j\geq 0}\beta_{ij}\phi^j(t)$.

So we have defined an isomorphism over k of affine varieties

$$\begin{array}{ccc}
\overline{x} & \mapsto & \Sigma_{j\geq 0}\alpha_j x_j \\
(\Sigma_{j\geq 0}\beta_{ij}\phi^j(t))_i & \leftarrow\!\shortmid & t
\end{array}$$

between $G_{\overline{a}}$ and the additive group; since the polynomials are additive, it is a group isomorphism.

Remark 0.4. The next step of the KSW argument involves considering the map induced via these isomorphisms from projecting out a co-ordinate, namely $\theta(t) = \Sigma_{i>0}\alpha'_i\Sigma_{j\geq 0}\beta_{ij}\phi^j(t)$ where $\overline{\alpha}'$ is obtained from $\overline{a}' := (a_1, \ldots, a_m)$. This turns out to be essentially $\wp$; I originally did these calculations in the expectation that this fact would naturally fall out, but in the end I don't see any way to shortcut the argument in KSW to obtain it.

–Martin Bays 2017