A REMARK ON GROUPS WITHOUT FINITE QUOTIENTS

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Abstract. We notice that the class of nontrivial groups without proper subgroups of finite index is not elementary, because some groups in this class, such as $\mathbb{Q} * \mathbb{Q}$, have ultrapowers that map homomorphically onto $\mathbb{Z}/p\mathbb{Z}$ for every prime p. Also, some ultrapowers of certain simple groups map homomorphically onto $\mathbb{Z}/2\mathbb{Z}$.

Definition. By NFQ we denote the class of nontrivial groups without proper subgroups of finite index (equivalently, nontrivial groups which have No nontrivial Finite Quotients).

For example, $(\mathbb{Q}, +)$ is NFQ.

Our main observation is the following proposition.

Main Proposition. *If A and B are NFQ groups, then the free product $G = A * B$ is NFQ as well; however, for every non-principal ultrafilter $\mathcal{U}$ on ω and for every prime p, there exists a homomorphism of $G^\omega/\mathcal{U}$ onto $\mathbb{Z}/p\mathbb{Z}$, and hence $G^\omega/\mathcal{U}$ and G^ω are not NFQ.*

Corollary. *The class NFQ is not elementary and is not closed under infinite products.*

Definition. A generating subset S of a group G is said to generate G *in n steps* if

$$G = \underbrace{(S^{\pm 1} \cup \{1\}) \cdots (S^{\pm 1} \cup \{1\})}_{n \text{ times}}.$$

For a group G and a group word $w = w(\bar{X})$, define

$$\mathrm{V}_w(G) = \{\, w(\bar{g}) \mid \bar{g} \subset G \,\}.$$

For example, $\mathrm{V}_{X^n}(G) = \{\, g^n \mid g \in G \,\}$, $\mathrm{V}_{[X,Y]}(G) = \{\, [g,h] \mid g, h \in G \,\}$.

Definition. If $w = w(\bar{X})$ is a group word and G a group, the *verbal width of G with respect to w* is the minimal number of steps in which $\mathrm{V}_w(G)$ generates $\langle \mathrm{V}_w(G) \rangle$.

Remark 1. A group generated by its NFQ subgroups is NFQ itself.

Remark 2. The class NFQ is closed under taking homomorphic images, extensions, direct sums, and free products.

Remark 3. An abelian group G is NFQ if and only if it is divisible (for every prime p, G/pG is a vector space over the finite field $\mathbb{Z}/p\mathbb{Z}$, so if it is nontrivial, then it has an epimorphism onto $\mathbb{Z}/p\mathbb{Z}$).

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Remark 4. An arbitrary (Cartesian) product of abelian NFQ groups is NFQ.

Remark 5. If G is an NFQ group and $n \in \mathbb{N}$, then $\mathrm{V}_{X^n}(G) \cup \mathrm{V}_{[X,Y]}(G)$ generates G (the abelianization $G/\langle \mathrm{V}_{[X,Y]}(G)\rangle$ is divisible by Remark 3).

Lemma 1. *Let G be a group and p a prime number. If $\mathrm{V}_{X^p}(G) \cup \mathrm{V}_{[X,Y]}(G)$ does not generate G in finitely many steps, then for every non-principal ultrafilter $\mathcal{U}$ on ω, the ultrapower $G^\omega/\mathcal{U}$ maps homomorphically onto $\mathbb{Z}/p\mathbb{Z}$.*

Proof. Denote H the abelianization of $G^\omega/\mathcal{U}$. Choose $f \in G^\omega$ such that

$$(\forall n < \omega)\ \Big(f(n) \notin \mathrm{V}_{X^p}(G) \cdot \underbrace{\mathrm{V}_{[X,Y]}(G) \cdots \mathrm{V}_{[X,Y]}(G)}_{n \text{ times}}\Big).$$

Then f represents a nontrivial element of H/H^p, and hence H/H^p is a nontrivial vector space over $\mathbb{Z}/p\mathbb{Z}$ and has an epimorphism onto $\mathbb{Z}/p\mathbb{Z}$. □

Remark 6. Since every commutator is the product of 3 squares (e.g. $[X,Y] = (YX)^{-2} \cdot (YX^2Y^{-1}) \cdot Y^2$), in the case $p = 2$, the hypothesis of the lemma reduces to "$\mathrm{V}_{X^2}(G)$ does not generate G in finitely many steps."

Proof of the main proposition relies on the following remarkable result of Rhemtulla.

Theorem (Rhemtulla, 1967, [2])**.** *If $w = w(\bar{X})$ is a group word such that there exists a group H such that $\{1\} \neq \langle \mathrm{V}_w(H)\rangle \neq H$, and if A and B are two nontrivial groups of which at least one has order greater than 2, then the verbal subgroup $\langle \mathrm{V}_w(A * B)\rangle$ of $A * B$ is not generated by $\mathrm{V}_w(A * B)$ in finitely many steps.*

Proof of the main proposition. Clearly G is NFQ, see Remark 1.

Let $w = w(X,Y,Z) = X^p[Y,Z]$. By Rhemtulla's theorem, G is not generated by $\mathrm{V}_w(G)$ in finitely many steps. Since $\mathrm{V}_w(G) \supset \mathrm{V}_{X^p}(G) \cup \mathrm{V}_{[Y,Z]}(G)$, $G^\omega/\mathcal{U}$ maps homomorphically onto $\mathbb{Z}/p\mathbb{Z}$ by Lemma 1. □

Another (more complicated) way to prove that NFQ is not a first-order property, without using Rhemtulla's theorem, is to consider the simple groups constructed in [1]: those groups are of infinite width with respect to X^2, and hence have ultrapowers that map homomorphically onto $\mathbb{Z}/2\mathbb{Z}$.

References

[1] A. Muranov, *Finitely generated infinite simple groups of infinite square width and vanishing stable commutator length*, `arXiv:0909.2294`, preprint, 41 pages, 2009.

[2] A. H. Rhemtulla, *A problem of bounded expressibility in free products*, Proc. Cambridge Philos. Soc. **64** (1968), 573–584.

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