EXPONENTIATIONS OVER THE QUANTUM ALGEBRA $U_q(sl_2(\mathbb{C}))$

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Abstract:

We define and compare, by model-theoretical methods, some exponentiations over the quantum algebra $U_q(sl_2(\mathbb{C}))$, for any parameter q. We discuss two cases, according to whether the parameter q is a root of unity.

MOTIVATIONS, SETTING and AIMS

Our setting

Quantum algebras are very interesting objects which are beginning to be investigated from a model theoretic point of view. This is witnessed, for instance, by [Zi] and [HL].

Motivations

This work is inspired by [LMP] where some possible exponentiations are defined over the universal enveloping U of the Lie algebra $sl_2(\mathbb{C})$ of 2x2 traceless matrices with entries in the field of complex numbers $\mathbb{C}$, via its finite-dimensional representations.

Aims

Our present aim is to define in a similar way some exponentiations over the quantum algebras $U_q := U_q(sl_2(\mathbb{C}))$, which can be regarded as the quantized version of U, for any parameter $q \in \mathbb{C} - \{0\}$, $q^2 \neq 1$.

Quantum algebra U_q

Consider any element $q \in \mathbb{C} - \{0\}$ such that $q^2 \neq 1$, the *quantum algebra* U_q described (see $[J]$, $[K]$) as the associative $\mathbb{C}$-algebra with generators $K,\ K^{-1},\ E,\ F$ and relations:

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ \ [E,F] = \frac{K-K^{-1}}{q-q^{-1}}. \qquad (1)$$

The relations (1) imply by induction for every integers s and t, $s,\ t \geq 2$, that:

$$[E,F^t] = [t]F^{t-1}\frac{Kq^{1-t}-K^{-1}q^{t-1}}{q-q^{-1}}, \qquad [E^s,F] = [s]E^{s-1}\frac{Kq^{s-1}-K^{-1}q^{1-s}}{q-q^{-1}},$$

where, for every $a \in \mathbb{Z}$, $[a] := \frac{q^a-q^{-a}}{q-q^{-1}}$ denotes the q-number of a.

Our strategy

We will discuss two cases, according to whether the parameter q is a root of unity. To define some exponentiations over U_q, we use:

- its simple representation maps,
- the natural matrix exponential map exp in $M_\ell(\mathbb{C})$, where

exp : $M_\ell(\mathbb{C}) \to GL_\ell(\mathbb{C})$, $\ell \in \omega - \{0\}$ is defined for any $A \in M_\ell(\mathbb{C})$ as the power series

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} .$$

Properties of U_q

Some properties of U_q, used for our results are the following (see $[J]$, $[K]$).

- U_q, as graded $\mathbb{C}$-algebra over the set of integers $\mathbb{Z}$, decomposes as

$$U_q = \oplus_{m\in\mathbb{Z}} U_{q,m},$$

where $U_{q,m} =< E^i.K^z.F^j : i-j=m,\ \ i,j\in\omega, z\in\mathbb{Z} >$ denotes the m-homogenous component of U_q.

- $U_{q,0}$ is isomorphic to the polynomial ring $k[C_q, K, K^{-1}]$, where $C_q := \frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2} + EF$ denotes the (quantum) Casimir element of U_q.
- Any element of $U_{q,m}$, for any q, can be written as $E^m u$, for $m \geq 0$, and uF^{-m}, for $m < 0$, with $u \in U_{q,0}$.

MAIN RESULTS

q is not a root of unity

In this case, all simple finite dimensional representations of U_q are classified in terms of highest weight. So, various exponentiations over U_q can be defined just by strategies similar to the ones used in [LMP] for the classical case.

$\forall\lambda \in \omega - \{0\}$, there exist (up to $\cong$) exactly two simple representations of dimension $\lambda+1$, denoted by $V_{\epsilon,\lambda}$, where $\epsilon \in \{-1,1\}$. Let $\Theta_{\varepsilon,,\lambda}$ the representation map: $\Theta_{\varepsilon,\lambda} : U_q \to End(V_{\varepsilon,\lambda})$, where $End(V_{\varepsilon,\lambda}) = M_{\lambda+1}(\mathbb{C})$.

New exponentiations

- We define a new exponential map over U_q: $\mathrm{EXP}_{\varepsilon,\lambda} : U_q \xrightarrow{\Theta_{\varepsilon,\lambda}} M_{\lambda+1}(\mathbb{C}) \xrightarrow{\exp} GL_{\lambda+1}(\mathbb{C})$, defined $\forall u \in U_q$ as:

$$\mathrm{EXP}_{\epsilon,\lambda}(u) := \exp(\Theta_{\epsilon,\lambda}(u)), \quad (\text{for } \epsilon \pm 1).$$

- Let $\mathcal{U}$ be a non principal ultrafilter on ω.

We define another exponential map: $\mathrm{EXP} : U_q \xrightarrow{[\Theta_{\varepsilon,\lambda}]} \prod_{\mathcal{U}} M_{\lambda+1}(\mathbb{C}) \xrightarrow{\exp} \prod_{\mathcal{U}} GL_{\lambda+1}(\mathbb{C})$, defined $\forall u \in U_q$ as:

$$\mathrm{EXP}(u) := [\mathrm{EXP}_{\epsilon,\lambda}(u)], \quad (\text{for } \epsilon \pm 1).$$

Our results

- For any $\lambda \in \omega$, $(U_q, \mathrm{EXP}_{\epsilon,\lambda}, GL_{\lambda+1}(\mathbb{C}))$ is a (non commutative) exponential ring.
- $(U_q, \mathrm{EXP}, \prod_{\mathcal{U}} GL_{\lambda+1}(\mathbb{C}))$ is a (non commutative) exponential ring.

We use the transfer the following properties of the classical matrix exponential to $\mathrm{EXP}_{\epsilon,\lambda}$. If $u,\ v \in U_q$, then $\forall\lambda \in \omega - \{0\}$:

(i) $\mathrm{EXP}_{\epsilon,\lambda}(0_{U_q}) = I_\lambda$, where 0_{U_q} denotes the identity element in U_q.

(ii) $\mathrm{EXP}_{\epsilon,\lambda}(u)\mathrm{EXP}_{\epsilon,\lambda}(-u) = I_\lambda$;

(iii) for u and v commuting, $\mathrm{EXP}_{\epsilon,\lambda}(u+v) = \mathrm{EXP}_{\epsilon,\lambda}(u)\mathrm{EXP}_{\epsilon,\lambda}(v)$;

(iv) for an invertible element v in U_q, $\mathrm{EXP}_{\epsilon,\lambda}(vuv^{-1}) = \Theta_{\epsilon,\lambda}(v)\mathrm{EXP}_{\epsilon,\lambda}(u)\Theta_{\epsilon,\lambda}(v)^{-1}$;

- $\forall\lambda \in \omega - \{0\}$, the map $\mathrm{EXP}_{\epsilon,\lambda}$ is surjective.
- For every non-principal ultrafilter $\mathcal{U}$ on ω, the map $[\Theta_{\varepsilon,\lambda}]$ is injective.

q is a root of unity

Assume that q is a primitive ℓ^{th} root of unity for $\ell \geq 3$.

There exists two families of simple representations of dimension ℓ, $V_{a,bc}$ depending on $a,b,c \in \mathbb{C}$ and $\tilde{V}_{d,f}$ depending on $f,d \in \mathbb{C}$.

We denote the related representation maps by $\Theta_{a,b,c} : U_q \to M_\ell(\mathbb{C})$ and $\widetilde{\Theta}_{d,f} : U_q \to M_\ell(\mathbb{C})$.

New exponentiations

- We define two new exponential maps over U_q:

$\mathrm{EXP}_{a,b,c} : U_q \xrightarrow{\Theta_{a,b,c}} M_\ell(\mathbb{C}) \xrightarrow{\exp} GL_\ell(\mathbb{C})$ and $\widetilde{\mathrm{EXP}}_{d,f} : U_q \xrightarrow{\widetilde{\Theta}_{d,f}} M_\ell(\mathbb{C}) \xrightarrow{\exp} GL_\ell(\mathbb{C})$, defined respectively $\forall u \in U_q$ as:

$$\mathrm{EXP}_{a,b,c}(u) := \exp(\Theta_{a,b,c}(u)), \qquad \widetilde{\mathrm{EXP}}_{d,f}(u) := \exp(\widetilde{\Theta}_{d,f}(u)).$$

- Then we will vary these maps along certain non principal ultrafilters $\mathcal{W}$ on ω^2.

We define other exponential maps, $\mathrm{EXP} : U_q \xrightarrow{[\Theta_{a,b,c}]} \prod_{\mathcal{W}} M_\ell(\mathbb{C}) \xrightarrow{\exp} \prod_{\mathcal{W}} GL_\ell(\mathbb{C})$ and $\widetilde{\mathrm{EXP}} : U_q \xrightarrow{[\widetilde{\Theta}_{,f}]} \prod_{\mathcal{W}} M_\ell(\mathbb{C}) \xrightarrow{\exp} \prod_{\mathcal{W}} GL_\ell(\mathbb{C})$, defined $\forall u \in U_q$ as:

$$\mathrm{EXP}(u) := [\mathrm{EXP}_{a,b,c}(u)], \qquad \widetilde{\mathrm{EXP}}_{d,f}(u) := \exp(\widetilde{\Theta}_{d,f}(u)).$$

Our results

Let $\mathcal{W}$ be a non-principal ultrafilter on ω^2 which will index subsets of complex numbers of the form (d_n, f_m) with $|f_m| > 1$, or (b_n, c_m) with $a_n.b_n$ a real constant and $|c_m| > 1$. For $u \in U_q$, set $\widetilde{\Theta}_{n,m}(u) := \Theta_{d_n,f_m}(u_z)$ and $\Theta_{n,m}(u) := \Theta_{a_n,b_n,c_m}(u_z)$. We prove:

- Let D, I be two countable bounded subsets of complex numbers of modulus strictly bigger than 1. For any $u \in \sum_{m\geq 0} U_{q,m} - \{0\}$, $\exists W_u \in \mathcal{W}$ such that for all $(n,m) \in W_u$ we have $\widetilde{\Theta}_{n,m}(u) \neq 0$.
- For any $u \in U_q - \{0\}$, $\exists W_u \in \mathcal{W}$ such that for all $n \in W_u$ we have $\Theta_{n,m}(u) \neq 0$.
- $(U_q, \mathrm{EXP}, \prod_{\mathcal{W}} GL_\ell(\mathbb{C}))$ and $(U_q, \widetilde{\mathrm{EXP}}, \prod_{\mathcal{W}} GL_\ell(\mathbb{C}))$ are exponential rings.

Recall that the classical enveloping algebra U is generated by X, Y, H and defining relations $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = H$. Consider $q_\ell = e^{\theta_\ell}$, where $\theta_\ell = \frac{2\pi.i}{\ell}$ and a non-principal ultraproduct of U_{q_ℓ}, $\ell \in \omega$, over a non principal ultrafilter $\mathcal{U}$ over ω.

- The map $\tau : U \to \prod_{\mathcal{U}} U_{q_\ell}$ sending X to $[E]_{\mathcal{U}}$, Y to $[F]_U$ and H to $[\frac{K-K^{-1}}{q-q^{-1}}]_U$ is injective.

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