Model Theory of Compact Complex Manifolds with an Automorphism

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Paper

"Model Theory of Compact Complex Manifolds with an Automorphism"
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Meromorphic dynamics

Let X be a compact complex manifold, and let $f : X \to X$ be a a dominant meromorphic map, e.g. $f \in \mathrm{Aut}(X)$. The pair (X, f) is called a *meromorphic dynamics* on X.

Examples

- $(\mathbb{P}^1(\mathbb{C}), \mathrm{id})$;
- $(T, x \mapsto x + a)$, where $T = \mathbb{C}^n/\Gamma$ a complex torus, $a \in T$;
- $(T, x \mapsto 2x)$.

The Chatzidakis-Hrushovski model theory of ACFA has proven effective for algebraic dynamics, the case of algebraic X.

CCM

Definition

Let $\mathcal{A}$ be the structure with a sort for each compact complex manifold X, and a relation for each complex analytic subset of a product of sorts.
CCM $:=$ Th($\mathcal{A}$).

Fact (Zilber)

CCM *eliminates quantifiers, and is* ω*-stable of finite rank.*

CCMA

Let $L_{\text{CCMA}} := L_{\text{CCM}} \cup \{\sigma\}$.

Definition

CCMA is the theory of the existentially closed models of

$$\text{CCM}_\forall \cup \{\sigma \text{ is an automorphism (on each sort)}\}.$$

If (X, f) is a meromorphic dynamics, define

$$(X, f)^\sharp := \{x \in X | \sigma(x) = f(x)\}.$$

σ-Varieties

To analyse meromorphic dynamics, we must first slightly generalise the notion:

Definition

A σ-variety (over $\mathbb{C}$) is a pair $\mathbb{X} = (X, \Gamma)$ where X is an irreducible analytic space and $\Gamma \subset X \times X$ is an irreducible closed analytic subset which is the graph of a generically finite-to-finite correspondence

$$\begin{array}{ccc} & \Gamma & \\ \swarrow & & \searrow \\ X & & X \end{array} .$$

A (relative) σ-variety over a σ-variety (Y, Γ_Y) is a dominant meromorphic map $(X, \Gamma_X) \to (Y, \Gamma_Y)$.
Define $(X, \Gamma)^\sharp$ as above:

$$(X, \Gamma)^\sharp := \{x \in X | (x, \sigma(x)) \in \Gamma\}.$$

Minimality and Analysis

Definition

A σ-variety $\mathbb{X}$ is *minimal* unless there exists a proper non-trivial relative σ-subvariety $\mathbb{Z} \to \mathbb{Y}$ of a (non-trivial) base-change $\mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$.

Analysis: any σ-variety $\mathbb{X} \to \mathbb{Y}$ admits an *analysis* in terms of minimal σ-varieties. Roughly, up to base changes, this is a resolution

$$\mathbb{X} = \mathbb{Z}_0 \to \mathbb{Z}_1 \to \cdots \to \mathbb{Z}_n = \mathbb{Y}$$

where $\mathbb{Z}_i \to \mathbb{Z}_{i+1}$ is minimal.

Zilber Trichotomy

Theorem (BHM)

For any minimal σ-variety $\mathbb{X} = (X, \Gamma)$, one of the following holds:

field type: $\mathbb{X}$ *is in finite-to-finite correspondence with* $(\mathbb{P}^1, \mathrm{id})$,

OR *group type:* $\mathbb{X}$ *is in finite-to-finite correspondence with* (T, Γ) *where* T *is a complex torus or* $\mathbb{G}_m^n$, *and* Γ *is a subgroup of* T^2, *and* (T, Γ) *is not of field type,*

OR *trivial: "*$\mathbb{X}$ *admits only binary relations": even after base change, there is no* σ*-subvariety* $\mathbb{Z} \subseteq \mathbb{X}^3$ *such that* $\pi_i : \mathbb{Z} \to \mathbb{X}$ *is dominant with generic fibre of dimension less than* $\dim X$.

(Analogous statements hold for relative σ*-varieties.)*

Example

Let T be a simple non-abelian complex torus, let $\alpha \in T$. Then $(T, +\alpha)$ is minimal; it is of group type if $\alpha \in \mathrm{Tor}(T)$, else of trivial type.
Now consider a complex torus G which is an extension

$$0 \to T_1 \to G \to^{\pi} T_2 \to 0$$

of simple non-abelian complex tori T_i, and e.g. let $\alpha \in G$ be non-torsion but $\pi(\alpha) \in \mathrm{Tor}(T_2)$.
Then the analysis is

$$(G, +\alpha) \to (T_2, +\pi(\alpha)) \to (\mathbb{C}, \mathrm{id}),$$

and $(G, +\alpha) \to (T_2, +\pi(\alpha))$ is trivial but $(T_2, +\pi(\alpha))$ is of group type.