

Model Theory of Compact Complex Manifolds with an Automorphism

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“Model Theory of Compact Complex Manifolds with an Automorphism”

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Meromorphic dynamics

Let X be a compact complex manifold, and let $f : X \rightarrow X$ be a dominant meromorphic map, e.g. $f \in \text{Aut}(X)$. The pair (X, f) is called a *meromorphic dynamics* on X .

Examples

- ▶ $(\mathbb{P}^1(\mathbb{C}), \text{id})$;
- ▶ $(T, x \mapsto x + a)$, where $T = \mathbb{C}^n/\Gamma$ a complex torus, $a \in T$;
- ▶ $(T, x \mapsto 2x)$.

The Chatzidakis-Hrushovski model theory of ACFA has proven effective for algebraic dynamics, the case of algebraic X .

Definition

Let \mathcal{A} be the structure with a sort for each compact complex manifold X , and a relation for each complex analytic subset of a product of sorts.

$\text{CCM} := \text{Th}(\mathcal{A})$.

Fact (Zilber)

CCM eliminates quantifiers, and is ω -stable of finite rank.

CCMA

Let $L_{\text{CCMA}} := L_{\text{CCM}} \cup \{\sigma\}$.

Definition

CCMA is the theory of the existentially closed models of

$\text{CCM}_{\forall} \cup \{\sigma \text{ is an automorphism (on each sort)}\}$.

If (X, f) is a meromorphic dynamics, define

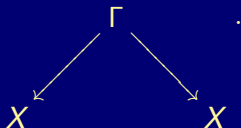
$$(X, f)^{\#} := \{x \in X \mid \sigma(x) = f(x)\}.$$

σ -Varieties

To analyse meromorphic dynamics, we must first slightly generalise the notion:

Definition

A σ -variety (over \mathbb{C}) is a pair $\mathbb{X} = (X, \Gamma)$ where X is an irreducible analytic space and $\Gamma \subset X \times X$ is an irreducible closed analytic subset which is the graph of a generically finite-to-finite correspondence



A (relative) σ -variety over a σ -variety (Y, Γ_Y) is a dominant meromorphic map $(X, \Gamma_X) \rightarrow (Y, \Gamma_Y)$. Define $(X, \Gamma)^\sharp$ as above:

$$(X, \Gamma)^\sharp := \{x \in X \mid (x, \sigma(x)) \in \Gamma\}.$$

Minimality and Analysis

Definition

A σ -variety \mathbb{X} is *minimal* unless there exists a proper non-trivial relative σ -subvariety $\mathbb{Z} \rightarrow \mathbb{Y}$ of a (non-trivial) base-change $\mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}$.

Analysis: any σ -variety $\mathbb{X} \rightarrow \mathbb{Y}$ admits an *analysis* in terms of minimal σ -varieties. Roughly, up to base changes, this is a resolution

$$\mathbb{X} = \mathbb{Z}_0 \rightarrow \mathbb{Z}_1 \rightarrow \cdots \rightarrow \mathbb{Z}_n = \mathbb{Y}$$

where $\mathbb{Z}_i \rightarrow \mathbb{Z}_{i+1}$ is minimal.

Zilber Trichotomy

Theorem (BHM)

For any minimal σ -variety $\mathbb{X} = (X, \Gamma)$, one of the following holds:

- field type: \mathbb{X} is in finite-to-finite correspondence with $(\mathbb{P}^1, \text{id})$,*
- OR group type: \mathbb{X} is in finite-to-finite correspondence with (T, Γ) where T is a complex torus or \mathbb{G}_m^n , and Γ is a subgroup of T^2 , and (T, Γ) is not of field type,*
- OR trivial: “ \mathbb{X} admits only binary relations”: even after base change, there is no σ -subvariety $\mathbb{Z} \subseteq \mathbb{X}^3$ such that $\pi_j : \mathbb{Z} \rightarrow \mathbb{X}$ is dominant with generic fibre of dimension less than $\dim X$.*

(Analogous statements hold for relative σ -varieties.)

Example

Let T be a simple non-abelian complex torus, let $\alpha \in T$. Then $(T, +\alpha)$ is minimal; it is of group type if $\alpha \in \text{Tor}(T)$, else of trivial type.

Now consider a complex torus G which is an extension

$$0 \rightarrow T_1 \rightarrow G \xrightarrow{\pi} T_2 \rightarrow 0$$

of simple non-abelian complex tori T_i , and e.g. let $\alpha \in G$ be non-torsion but $\pi(\alpha) \in \text{Tor}(T_2)$.

Then the analysis is

$$(G, +\alpha) \rightarrow (T_2, +\pi(\alpha)) \rightarrow (\mathbb{C}, \text{id}),$$

and $(G, +\alpha) \rightarrow (T_2, +\pi(\alpha))$ is trivial but $(T_2, +\pi(\alpha))$ is of group type.