Geometric stability theory and pseudofinite combinatorics

Martin Bays

17.09.2020
Chemnitz

Erdős geometry

Example (Szemerédi-Trotter (1983))

Given N^2 points and N^2 lines in $\mathbb{R}^2$, the number of incidences is bounded as

$$|\{(p,l) : p \in l\}| \leq O(N^{\frac{8}{3}}).$$

Example ("Sum-product phenomenon")

For any finite set $A \subseteq \mathbb{C}$,

$$|A| \leq O(\max(|A+A|, |A*A|)^{\frac{4}{5}}).$$

(This particular bound is due to Solymosi (2005).)

Example (Orchard problem)

Find large finite subsets $X \subseteq \mathbb{R}^2$ such that $\geq c|X|^2$ lines contain at least 3 points of X.

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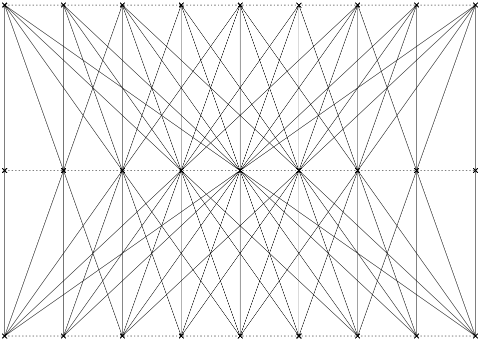
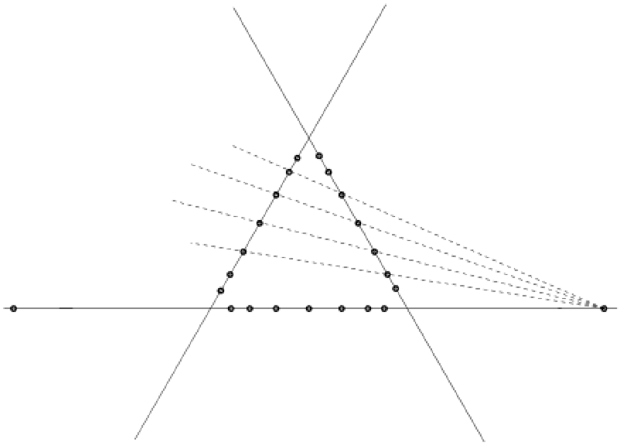


$(\sim \frac{|X|^2}{18}$ 3-point lines)

Orchard solution: linear



(Image from Elekes-Szabó "On triple lines and cubic curves")

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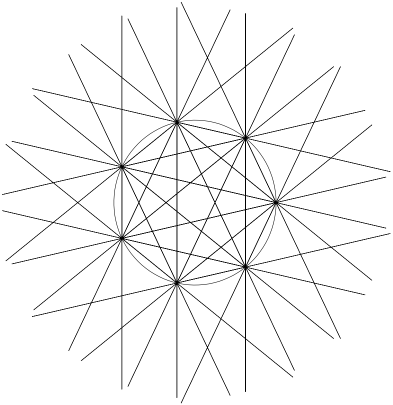
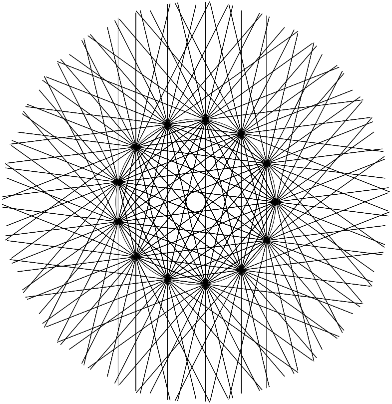
Orchard solution: multiplicative N=7

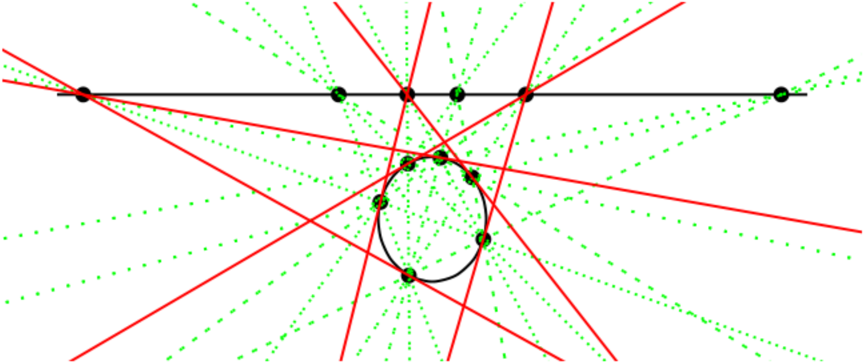


Orchard solution: multiplicative N=13



($\sim \frac{|X|^2}{8}$ 3-point lines)

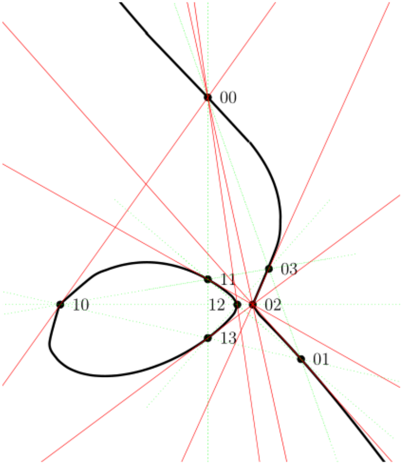
Orchard solution: multiplicative transformed



(Image from Green-Tao "On sets defining few ordinary lines")

($\sim \frac{|X|^2}{8}$ 3-point lines)

Orchard solution: elliptic



(Image from Green-Tao "On sets defining few ordinary lines")

($\sim \frac{|X|^2}{6}$ 3-point lines)

Orchard solutions

Cubic curves provide solutions to the orchard problem. Conversely:

Theorem (Elekes-Szabó '13)

Let $C \subseteq \mathbb{R}^2$ be an irreducible algebraic curve which is **not** *cubic, i.e.* $\deg(C) \neq 3$.
Then for $X \subseteq_{\text{fin}} C(\mathbb{R})$,

$$|\{ \textit{3-point lines} \}| \leq O(|X|^{2-\epsilon}),$$

where $\epsilon = \epsilon(\deg(C)) > 0$.

Structures

- A **structure** is a set M with a choice of $\emptyset$-**definable sets** $X \subseteq M^n$, closed under intersection, complement, cartesian product, and co-ordinate projection, and including the diagonal $\Delta \subseteq M^2$.
- Examples:
  (i) Pure infinite set:
  $\emptyset$-definable sets are boolean combinations of diagonals.
  (ii) Vector space over a division ring:
  $\emptyset$-definable sets are boolean combinations of linear subspaces.
  (iii) Algebraically closed field:
  $\emptyset$-definable sets are boolean combinations of algebraic sets over the prime field.
- The M-**definable sets** are those of form $\{x : (x, m) \in X\} \subseteq M^n$ where $X \subseteq M^{n+m}$ is $\emptyset$-definable and $m \in M^m$.
- We consider only structures M which are ω_1-**compact**:
  if $X_0 \supseteq X_1 \supseteq \ldots$ is a decreasing chain of non-empty M-definable sets, then $\bigcap_{i\in\omega} X_i \neq \emptyset$.

Geometric stability theory: minimality

- An infinite $\emptyset$-definable set X is **minimal** if the only M-definable subsets are the finite subsets and their complements.
- Then for $C \subseteq X$, the **algebraic closure** $\operatorname{acl}(C)$ is the closure of C under $\emptyset$-definable finitely valued multifunctions $X^n \to X$.
- This induces a **dimension function** $\dim(C)$.

Examples

(i) Pure infinite set:
  - $\operatorname{acl}(C) = C$.
  - $\dim(C) = |C|$.

(ii) Vector space over a division ring k:
  - $\operatorname{acl}(C) = \langle C \rangle_k$.
  - $\dim(C) = \dim_k(\langle C \rangle_k)$.

(iii) Algebraically closed field:
  - $\operatorname{acl}(C) =$ [algebraically closed subfield generated by C].
  - $\dim(C) = \operatorname{trd}(C)$.

Combinatorial geometries

Geometry of a minimal set X:

$$\mathcal{G}_X := (\{\mathrm{acl}(x) : x \in X\}; \mathrm{acl}).$$

Definitions

A geometry $(P; \mathrm{cl})$ is **modular** if for $a, b \in P$ and $C = cl(C) \subseteq P$, if $a \in \mathrm{cl}(bC)$ then $a \in \mathrm{cl}(bc)$ for some $c \in C$.

Fact (Veblen-Young co-ordinatisation theorem)

A geometry is modular if and only if it is the disjoint union of

- *geometries of dimension* ≤ 3*, and*
- *projective geometries* $\mathbb{P}_k(V)$ *of vector spaces over division rings.*

Trichotomy

Theorem (Zilber's weak trichotomy theorem; 1980's)

For X minimal, up to naming parameters, exactly one of the following holds:

(i) **Modular and disintegrated**:
For $A \subseteq \mathcal{G}_X$, $\mathrm{acl}(A) = A$.

(ii) **Modular and not disintegrated**:
$\mathcal{G}_X = \mathbb{P}_k(V)$
where V is a definable abelian group with a division ring k of definable finitely-valued endomorphisms and no further structure, and X is in definable finite-to-finite correspondence with V.

(iii) **Not modular**:
There exists a 2-dimensional definable family of minimal subsets of X^2, e.g. $\{\{y = ax + b\} : a, b\}$.

Coherence

- Let K be a field.
- Let $V \subseteq K^m$ be an algebraic set over K.
- **"Trivial bound"**: For $A_i \subseteq K$ with $|A_i| = N$, we have

$$\left| V \cap \prod_{i=1}^{m} A_i \right| \leq O(N^{\dim(V)}).$$

- Say V is **coherent** if the exponent in the trivial bound is optimal i.e. for no $\epsilon > 0$ do we have for $A_i \subseteq K$ with $|A_i| = N$

$$\left| V \cap \prod_{i=1}^{m} A_i \right| \leq O(N^{\dim(V)-\epsilon}).$$

Coherence examples

- $V := \{(x,y,a,b) : y = ax + b\}$; $\dim(V) = 3$.
  By Szemerédi-Trotter, for $K = \mathbb{R}$ (in fact: whenever $\mathrm{char}(K) = 0$), if $|A_i| = N$ then

  $$|V \cap \prod_{i=1}^{4} A_i| \leq O(N^{\frac{8}{3}}) = O(N^{3-\frac{1}{3}}),$$

  so V is not coherent.
- Sum-product implies $V := \{(x,y,z,w) : z = x + y, w = xy\} \subseteq \mathbb{C}^4$ is not coherent.
- Orchard: Given an irreducible algebraic curve $C \subseteq \mathbb{C}^2$, let
  $V_C := \{(x,y,z) \in C^3 : x,y,z \text{ are collinear and distinct } \} \subseteq \mathbb{C}^6$.

  Then by Elekes-Szabó, C is coherent iff cubic.

Positive characteristic

For $K = \mathbb{F}_p^{\mathrm{alg}}$, **any** algebraic set $V \subseteq K^n$ is coherent:
in fact there is $r > 0$ such that for $n \gg 0$,

$$|V(\mathbb{F}_{p^n})| \geq r(p^n)^{\dim V}.$$

Modularity of coherence

- Szemerédi-Trotter for $\mathbb{C}$ implies:
  The family of lines on the plane $\{y = ax + b\} \subseteq \mathbb{C}^4$ is not coherent.
- Generalisations imply:
  no ≥ 2-dimensional family of plane curves $C_b \subseteq \mathbb{C}^2$ is coherent.
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- Elekes-Szabó '12: using these Szemerédi-Trotter bounds and arguments inspired by model theory (group configuration), characterise coherence for surfaces $V \subseteq \mathbb{C}^3$.
- B-Breuillard '18: associate a modular geometry to coherent structure, and hence characterise coherence for $V \subseteq \mathbb{C}^n$.

Geometry of coherence

Fix $\mathcal{U} \subseteq \mathbb{P}(\mathbb{N})$ a non-principal ultrafilter, and let $K := \mathbb{C}^{\mathcal{U}}$ be the corresponding countable ultrapower of $\mathbb{C}$. Let $\mathcal{U}'$ be a further ultrafilter and set $\mathbb{K} := K^{\mathcal{U}'}$. Fix $N_i \in \mathbb{N}$.

Definition (Hrushovski-Wagner coarse pseudo-finite dimension)

For $\overline{a} \in \mathbb{K}^n$, define $\boldsymbol{\delta}(\overline{a}) \in [0, \infty]$ by:
$\boldsymbol{\delta}(\overline{a}) \leq \alpha \in \mathbb{R}$ if and only if $\overline{a} \in (\prod_{i \to \mathcal{U}} A_i)^{\mathcal{U}'}$ for some $A_i \subseteq_{\text{fin}} \mathbb{C}^n$ with $|A_i| \leq O(N_i^{\alpha})$.

- Say $P \subseteq \mathbb{K}$ is **coherent** if $\boldsymbol{\delta}(\overline{a}) = \operatorname{trd}(\mathbb{C}(\overline{a})/\mathbb{C})$ for any $\overline{a} \in P^{<\omega}$.
- Then an irreducible algebraic set $V \subseteq \mathbb{C}^n$ is coherent iff it is the $\mathbb{C}$-Zariski closure of some $\overline{a} \in P^n$ for some coherent P (for some choice of $\mathcal{U}'$ and N_i).

Lemma (B-Breuillard '18)

If $P \subseteq \mathbb{K}$ is a maximal coherent subset, then field-theoretic algebraic closure on P is a <u>modular</u> geometry $(P; \operatorname{acl})$.

Characterising coherence

- A **special subgroup** H is an algebraic subgroup of a power of a 1-dimensional algebraic group, $H \leq G^n$.
- A variety $V \subseteq \mathbb{C}^n$ is **special** if it is in co-ordinatewise algebraic correspondence with a product of special subgroups.

Theorem (B-Breuillard '18)

$V \subseteq \mathbb{C}^n$ is coherent if and only if it is special.

- (For a surface $V \subseteq \mathbb{C}^3$, this was already proven by Elekes-Szabó (2012)).

Generalised sum-product

Corollary (B-Breuillard '18)

If $*_1, *_2 : \mathbb{C}^2 \to \mathbb{C}$ are (induced from) group operations on 1-dimensional algebraic groups G_i (i.e. $\mathbb{G}_a$ or $\mathbb{G}_m$ or an elliptic curve), then either G_1 and G_2 are isogenous,
or there exist $c, \epsilon > 0$ such that for finite sets $A \subseteq_{\text{fin}} \mathbb{C}$,

$$|A| \leq c \cdot (\max(|A *_1 A|, |A *_2 A|)^{1-\epsilon}).$$

Higher dimension

Question (Higher orchard)

Which algebraic surfaces $S \subseteq \mathbb{R}^3$ *support arbitrarily large finite subsets* $X \subseteq S$ *with* $\geq c|X|^2$ *3-point lines?*

Question (Erdős discrete distances problem)

Given N *points in* $\mathbb{R}^2$*, what is the minimal number of distances between pairs of the points? (Guth-Katz '15:* $\geq c\frac{N}{\log N}$*.)*

General context: rather than $V \subseteq \mathbb{C}^n$, consider subvarieties $V \subseteq \prod_i W_i$ where $W_1, \ldots, W_n$ are arbitrary complex algebraic varieties.

Coherence with general position

$V \subseteq \prod_{i=1}^{n} W_i$, $\dim(W_i) = d$.

- V is **coherent** if for no $\epsilon > 0$ do we have a bound

$$\left| V \cap \prod_i A_i \right| \leq O\left(N^{\dim(V)-\epsilon} \right)$$

  for $A_i \subseteq W_i$ in "sufficiently general position" with $|A_i| \leq N^d$.
- A **special subgroup** H is an algebraic subgroup of a power of a <u>commutative</u> d-dimensional algebraic group, $H \leq G^k$
  - (and $H = \ker(M)^0$ for some $M \in \mathrm{Mat}_k(F)$ for some division ring F of quasi-endomorphisms.)
- A variety is **special** if it is in co-ordinatewise algebraic correspondence with a product of special subgroups.
- Generalising a result of [Elekes-Szabó '12] in the case $n = 3$:

Theorem (B-Breuillard '18)

V is coherent if and only if it is special.

General position

"Sufficiently general position" means (C, τ)-general position for some C, τ, where:

Definition

$A \subseteq_{\text{fin}} W$ is in (C, τ)-**general position** if for any proper subvariety $W' \subsetneq W$ of complexity $\leq C$, we have $|W' \cap A| \leq |A|^{\frac{1}{\tau}}$.

Pseudofinitely, general position corresponds to a "minimality" condition: $a \in W(\mathbb{K})$ is in (coarse) general position if

$$\forall B \subseteq \mathbb{K}. \ (\mathrm{trd}(a/B) < \mathrm{trd}(a) \Rightarrow \boldsymbol{\delta}(a/B) = 0).$$

Approximate subgroups of linear algebraic groups

Example (Approximate subgroups of nilpotent algebraic groups)

$$X := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \{-N, \dots, N\},\ c \in \{-N^2, \dots, N^2\} \right\}$$

then $|X^3 \cap \Gamma_*| \geq c|X|^2$,
but X is not in general position.

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but X is <u>not</u> in general position.

- Define "weak general position" (wgp) by $\mathrm{trd}(a/B) < \mathrm{trd}(a) \Rightarrow \boldsymbol{\delta}(a/B) < \boldsymbol{\delta}(a)$.
- By a result of Breuillard-Green-Tao '11: if G is a linear complex algebraic group, then $\Gamma_* \leq G$ is wgp-coherent iff G is nilpotent.
- Can we characterise wgp-coherence in terms of nilpotent algebraic groups?

Positive characteristic revisited

- For $K = \mathbb{F}_p^{\mathrm{alg}}$, **any** algebraic set $V \subseteq K^n$ is coherent.

Positive characteristic revisited

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Positive characteristic revisited

- For $K = \mathbb{F}_p^{\mathrm{alg}}$, **any** algebraic set $V \subseteq K^n$ is coherent.
- Hrushovski conjectures that coherence satisfies trichotomy in the form: "Any non-modularity of coherence is due to an infinite pseudofinite field".
- So what about coherence in K where $K \cap \mathbb{F}_p^{\mathrm{alg}}$ is finite, e.g. $K = \mathbb{F}_p(t)$?

Distal cutting

Definition

A **distal cell decomposition** of a binary relation $R \subseteq A \times B$ consists of relations $\Delta_1, \ldots, \Delta_t \subseteq A \times B^s$ such that:
for any finite $B_0 \subseteq_{\mathrm{fin}} B$, any $a \in A$ is in some $\Delta_i(b)$ with $b \in B_0^s$ such that for all $b' \in B_0$:
$\Delta_i(b) \subseteq R(b')$ or $\Delta_i(b) \cap R(b') = \emptyset$.

Theorem (Chernikov-Galvin-Starchenko, Chernikov-Starchenko '20; "Szemerédi-Trotter case")

If $R \subseteq A \times B$ admits a distal cell decomposition and

$$\exists t \in \mathbb{N}.\ \forall b \neq b' \in B.\ |R(b) \cap R(b')| < t,$$

then there is $\epsilon > 0$ such that for all N and $A_0 \subseteq A$ and $B_0 \subseteq B$ with $|A_0| \leq N^2, |B_0| \leq N^m$:

$$R \cap (A_0 \times B_0) \leq O(N^{m+1-\epsilon}).$$

Distality in $\mathbb{F}_p(t)$

Fact (Chernikov-Simon '12)

A theory is distal iff every definable relation admits a distal cell decomposition with definable Δ_i.

The fields $\mathbb{R}$ and $\mathbb{Q}_p$ are distal. $\mathbb{F}_p(t)$ is certainly not distal. However

Proposition (B - J-F Martin '20?)

If K is a valued field with finite residue field, then it is "quantifier-free distal": every quantifier-free definable relation admits a distal cell decomposition with quantifier-free definable Δ_i.

Corollary

If K is a finitely generated field of positive characteristic (e.g. $\mathbb{F}_p(t)$), then any polynomially defined relation $R \subseteq K^n \times K^m$ admits a distal cell decomposition.
Hence no 2-dimensional algebraic family of plane curves $V \subseteq K^2 \times K^m$ is coherent, and coherence in K is modular.

Thanks

Thanks.

Bonus: Speculation

Tentative Definition

- *$V \subseteq \prod_i W_i$ is* **special** *if there are $f_i : W_i \to S_i$ such that: $V' := \overline{(\prod f_i)(V)} \subseteq \prod S_i$ is special, and there are commutative group schemes $G_i \to S_i$ and a subgroup scheme $H \to V'$ of $\prod_i G_i \to \prod_i S_i$ (with fibres being subgroups defined by division rings) and a relative algebraic correspondence $V \sim H$ over V' projecting to relative correspondences $W_i \sim G_i$.*
- $\{(0, \ldots, 0)\} \subseteq \{0\} \times \ldots \times \{0\}$ *is special.*

- $\Gamma_G \subseteq G^3$ is special for G a nilpotent algebraic group.
- Coherent $\Leftrightarrow$ special?

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