A group action version of the Elekes-Szabó theorem

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1. Elekes-Szabó

Th^m [Elekes-Szabó, 1d version]:

- C alg^c curve, $\Gamma \subseteq C^3$ alg^c surface

$\dim(\pi_{ij}(\Gamma)) = 2$ $\quad \pi_{ij} : \begin{matrix} C\times C\times C \\ \downarrow \\ C\times C \end{matrix}$ i,j co-ords

char 0 $C, \Gamma / \bar{\mathbb{Q}} \subseteq \mathbb{C}$

- Suppose

$\circledast$ $\forall c, \eta > 0.\ \exists X \subseteq_{fin} C(\mathbb{C})\ |X^3 \cap \Gamma| > c|X|^{2-\eta}$

(e.g. $\Gamma = \{x+y=z\}$, $X = \{-N, \dots, N\}$)

Then $\exists (G,+)$ alg^c grp $\exists \alpha_i : C \rightsquigarrow G$ alg^c correspondences ($/\bar{\mathbb{Q}}$)

s.t. $\Gamma \sim \{\alpha_1(x) + \alpha_2(y) = \alpha_3(z)\}$

$\mathcal{U}$ non-princ^l ultrafilter on $\mathbb{N}$

Psf v^n 1: $\circledast \Leftrightarrow \circledast'$: $\overset{\text{infte}}{\hat{\exists}}$ psf $X \subseteq C(\mathbb{C}^{\mathcal{U}})$. $\delta_{|X|}(X^3 \cap \Gamma) = 2$

where $\delta_\xi(Y) := st(\log_\xi |Y|) \in \mathbb{R} \cup \{\pm\infty\}$

(for Y psf)

i.e. $\delta_{(\xi_i)_i/\mathcal{U}}((Y_i)_i/\mathcal{U}) = \lim_{i\to\mathcal{U}} \log_{\xi_i}|Y_i|$ in $\mathbb{R}$

$d(a) := \text{trd}(\bar{\mathbb{Q}}(a)/\bar{\mathbb{Q}})$

where $d(a/C) := \text{trd}(\bar{\mathbb{Q}}(a,C)/\bar{\mathbb{Q}}(C))$

Psf v^n 2: C alg^c curve, Let $\xi \in \mathbb{N}^{\mathcal{U}} \setminus \mathbb{N}$, "adequate language" $\mathcal{L}$.

$a_1, a_2, a_3 \in C(\mathbb{C}^{\mathcal{U}})$ gen^c (i.e. $d(a_i) = \dim C = 1$)

$a_i \perp a_j$ (i.e. $d(a_i/a_j) = d(a_i)$) $\quad \{i,j,k\} = \{1,2,3\}$

$a_i \in \text{acl}(a_j, a_k)$

$a_i \overset{\delta}{\perp} a_j$ (i.e. $\delta(a_i/a_j) = \delta(a_i)$ where

$\delta(a/C) := \delta_\xi(a/C) := \inf_{\substack{\text{psf } X \ni a \\ X\ \mathcal{L}\text{-def}^{ble}/C}} \delta_\xi(X)$

$0 < \delta(a_i) < \infty$

Fact: For any psf $(X_i)_{i\in\omega}$

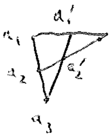
exists adequate $\mathcal{L} \supseteq \{+, \cdot, (X_i)_{i\in\omega}\}$

Then $\exists a_i' \in G$ alg^c gr^p/$\bar{\mathbb{Q}}$ $\dim G = 1$

$a_i' \sim a_i$ (i.e. $\text{acl}(a_i') = \text{acl}(a_i)$)

$a_1' + a_2' = a_3'$.

Pf idea:



$\text{cb}(a_2 a_2' / a_1 a_1') =: e$ $\quad d(e) = 1$ by Sz-T.

grp conf $\rightsquigarrow$ result. $\square$

Fact/Def^n: Sps $x \underset{g}{\sim} y$ and $x \perp g \perp y$.

Then $\text{acl}(\text{cb}(xy/g))$ is smallest $A = \text{acl}(A) \subseteq \text{acl}(g)$

s.t. $x \underset{A}{\sim} y$.

- Higher dim version:

Replace "$\dim C = 1$" with "a_i is in coarse general pos^n (cgp)"

meaning $0 < \delta(a_i) < \infty$ and $\delta(a_i/C) = 0$ if $d(a_i/C) < d(a_i)$

2. Recognising actions

Sps (G,X) is a faithful connected algebraic homogeneous space (f.c.a.h.s)

i.e. $(G;\cdot)$ connected alg^c grp, X var^y

$*: G\times X \to X$ morphism faithful transitive grp^p action.

(e.g. $(GL_n, \mathbb{A}^n)$)   (Rem^k: G ab^n $\Rightarrow X \cong G$)

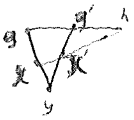
Let $(g,x) \in G\times X$ gen^c. Set $y := g*x$.

Then (i) $x \underset{g}{\sim} y$, $x \downarrow g \downarrow y$ $(d(y/g) = d(x/g) = d(x) = \dim G \geq d(y) \geq d(y/g))$

(ii) $x \downarrow y$ (by trans^y)

(iii) $cb(xy/g) \sim g$ (by faithfulness)

(iv) Let $x'g' \underset{y}{\overset{\downarrow}{\equiv}} xg$, let $h := cb(xx'/gg')$ $(h = g'^{-1}g)$



Then $d(h) = d(g)$.

[equiv $g \downarrow h \downarrow g'$]

Grp conf th^m ⇝ (i)-(iv) $\Rightarrow \exists (G,X) \ni (g',x')$ as above, $g \sim g'$, $x \sim x'$

3. (= 1. + 2.) A grp^p action ES th^m

Th^m [BZ '23]: $\delta = \delta^{\mathcal{L}}_{\xi}$, $\xi \in \mathbb{N}^{\mathcal{U}} \setminus \mathbb{N}$, $\mathcal{L}$ adequate

Sps
- x, y cgp
- g cgp (or just fLPgp & wgp)
- $x \underset{g}{\sim} y$ • $cb(xy/g) \sim g$
- $x \overset{\delta}{\downarrow} g \overset{\delta}{\downarrow} y$ $(\Rightarrow \delta(y) = \delta(x))$
- $acl(x) \cap acl(y) = acl(\emptyset)$ (else work over it)

Then: (I) exists (G,X) f.c.a.h.s

$x \underset{gen^c}{\sim} x' \in X$

(II) moreover, G ab^n

This week! → (III) Moreover, same conclusion as in ES:

$x \sim x'$, $g \sim g'$, $y \sim y'$, $(x',g',y') \underset{gen^c}{\in} \Gamma_*$

Def^n: g is wgp iff $0 < \delta(g) < \infty$ and $\delta(a/c) < \delta(a)$ iff $d(a/c) < d(a)$, i.e. $a \overset{\delta}{\downarrow} c \Rightarrow a \downarrow c$.

g is fLPgp iff $\delta(h) \geq d(h) \cdot \frac{\delta(g)}{d(g)}$ whenever $h \in acl(g_1,\dots,g_n)$, $g_i \overset{\delta}{\equiv} g$, $g_i \overset{\delta}{\downarrow} g_{<i}$

Fact: cgp $\Rightarrow$ fLPgp & wgp

Examples $X := \{-N^2,\dots,N^2\}$

$H := \{-N,\dots,N\}$

$\delta_N(X \times H \times X \cap \Gamma_+) = 3 = \delta(X) + \delta(H)$

⇝ $(x,g,\overset{\cup}{y})$ $\delta(x) = 2 = \delta(y)$, $\delta(g) = 1$

$\delta(x,g,y) = 3$ $x \overset{\delta}{\downarrow} g \overset{\delta}{\downarrow} y$

- Sim with $X := \{-N,\dots,N\} \times \{-\pi N,\dots,\pi N\}$

$H := \{-N,\dots,N\} \cup \{-\pi N,\dots,\pi N\}$

Cor^y: In ES, can weaken ⊛ to:

⊛ $\forall c,\eta > 0\ \exists X, H \underset{fin}{\subseteq} G(\mathbb{C})$. $|X*H*X \cap \Gamma| > c(|X||H|)^{1}$

Note: 1d case already improved on (with explicit η) by Raz-Sharir-de Zeeuw 2015.

Higher dim case (with cgp and LPgp) appears to be new.

Action ES

pf of Th^m: wLOG $\delta(g)=d(g)$

(I)

$h := cb(xx'/gg')$

$d(g) \underset{\text{f.d.gp}}{\le} d(h) \le \delta(h) \underset{\uparrow\, \delta \ge T \,+\, x \text{ cgp}}{\le} \delta(x)$

Replace $g \rightsquigarrow h$

$y \rightsquigarrow x'$

assumptions preserved

After finitely many iterations, $d(h)=d(g)$.

Also $x \downarrow y$.

So grp conf $\rightsquigarrow (G,X)$

(II) follows from:

Th^m [BDZ]: $(g,x) \in_{\text{gen}} (G,X)$ f.c.a.h.s, g wgp, x cgp

(x wgp $\Rightarrow$ G nilp was in BDZ'22)

$x \overset{\delta}{\downarrow} g \overset{\delta}{\downarrow} g*x$

Then G is ab^n. (so (G,X) is PHS)

pf: For some n, $G \curvearrowright X^n$ diag action has trivial stabs

$\bar{x} = (x_1,\ldots,x_n)$ s.t. $\bar{x} \overset{\delta}{\downarrow} g \overset{\delta}{\downarrow} g*\bar{x}$

$\delta(g) = \delta(g/g*\bar{x}) \underset{\uparrow\, \text{stab}_G(\bar{x})=\{1\}}{=} \delta(\bar{x}/g*\bar{x}) \le \delta(\bar{x}) = n\delta(x)$

Iterate as in (I) $(g \rightsquigarrow g'^{-1}\cdot g \ \ldots)$

and "take limit"

$\rightsquigarrow \delta(g) = \delta(g'^{-1}\cdot g) \le_n \delta(x)$

$\rightsquigarrow \delta=0$-approx subgrp

$\underset{\text{BGT}}{\rightsquigarrow}$ G nilp

e.g. 2-step nilp

$t := [g_1^{-1}g_2, g_3^{-1}g_4] \in Z(G)$ $(g_i)_i$ δ-indep $/x$

"action BSG" $\rightsquigarrow \delta(t*x) \le \delta(x)$

Then $\delta(x/t*x) \ge \delta(t*x/x) \overset{\swarrow\, G_x \cap Z(G)=1}{=} \delta(t/x) = \delta(t) > 0$ $\swarrow$ $(\bar{g} \not\downarrow t \Rightarrow \delta(t)=\delta(\bar{g})-\delta(\bar{g}/t)>0)$

but $d(x/t*x) \le$ $\dim(Z(G)*x) < \dim(X) = d(x)$ $\because_x$ cgp $\square$

(III) After iterating, $g \underset{y}{\sim} x$

Follows: also originally.

Then $x \downarrow y$: else, $\delta(g) = \delta(g/y) \equiv \delta(x/y) \underset{x \text{ cgp}}{=0}$ $\because$

Then $d(g)=d(g/y)=d(x/y)=d(x)=\dim G$

so $d(h)=d(g)$ $\rightsquigarrow$ grp conf immediately $\square$

(Note: too long for 1h, didn't get to pf...
If doing again: start with (III) statement
Similarly $\rightsquigarrow$ pf 1 $\rightsquigarrow$ pf 2, then pf ?)