Hrushovski-Weil

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Notes for a talk in a Münster seminar on stability theory.

1 Introduction

We aim now for the group configuration theorem. This powerful theorem shows that whenever one has a certain configuration of points in a stable theory satisfying certain independence and algebraicity conditions, it comes from a $\bigwedge$-definable group (or group action).

Here we prove a version of the "Hrushovski-Weil group chunk theorem", which recognises a $\bigwedge$-definable group (action) from a generically presented one. This will be the final step in the proof of the group configuration theorem.

2 Preliminaries

Let $\mathbb{M}$ be a monster model of a stable theory $T = T^{\mathrm{eq}}$.

Notation.

- If $p \in S(A)$ is stationary and $B \supseteq A$, $p|_B \in S(B)$ is the unique non-forking extension.
- If $p, q \in S(A)$ are stationary, their **product type** is $p \otimes q := \mathrm{tp}(b, c/A)$ where $c \vDash q$ and $b \vDash p|_{Ac}$, i.e. $(b, c) \vDash p \times q$ and $b \mathop{\smile\!\!\!\!\!\mid}_A c$.
- $p^{(2)} := p \otimes p$, $p^{(3)} := p \otimes p \otimes p$ etc.

For notational convenience, <u>assume</u> (by adding parameters) that $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$. So types over $\emptyset$ are stationary.

3 Germs

Definition. Let $\mathfrak{p}, \mathfrak{q} \in S(\mathbb{M})$.

Say a definable partial function f is **defined at** $\mathfrak{p}$ if $\mathfrak{p}(x) \vDash x \in \mathrm{dom}(f)$. The **germ of** f **at** $\mathfrak{p}$ is then the equivalence class $\widetilde{f}$ under the equivalence relation $\mathfrak{p}(x) \vDash f_1(x) = f_2(x)$.

Write $\widetilde{f} : \mathfrak{p} \to \mathfrak{q}$ if $\mathfrak{p}(x) \vDash \mathfrak{q}(f(x))$ for some (any) representative f (i.e. $f_*(\mathfrak{p}) = \mathfrak{q}$).

If $\widetilde{f} : \mathfrak{p} \to \mathfrak{q}$ has a representative f defined over b and $a \vDash \mathfrak{p}|_b$, let $\widetilde{f}(a) := f(a)$. This is well-defined (if g is another representative defined over b then $g(a) = f(a)$ since $a \vDash \mathfrak{p}|_b(x) \vDash f(x) = g(x)$), and $\widetilde{f}(a) \vDash \mathfrak{q}|_b$.

Example. In an o-minimal theory, let $\mathfrak{p}_{+\infty}$ be the global type of a positive infinite element. Then a definable function f which is unbounded on all end-segments $(a, +\infty)$ defines a germ $\widetilde{f} : \mathfrak{p}_{+\infty} \to \mathfrak{p}_{+\infty}$, and $\widetilde{f} = \widetilde{g}$ iff f and g are eventually equal.

Definition. $\mathrm{Aut}(\mathbb{M})$ acts on germs: given $\widetilde{f} : \mathfrak{p} \to \mathfrak{q}$ and $\sigma \in \mathrm{Aut}(\mathbb{M})$, $\widetilde{f}^\sigma := \widetilde{(f^\sigma)} : \mathfrak{p}^\sigma \to \mathfrak{q}^\sigma$ does not depend on the choice of f.

A (possibly infinite) tuple $\bar{b}$ is a **code** for $\widetilde{f}$ if $\forall \sigma \in \mathrm{Aut}(\mathbb{M}).(\bar{b} = \bar{b}^\sigma \leftrightarrow \widetilde{f} = \widetilde{f}^\sigma)$; then set $\ulcorner \widetilde{f} \urcorner := \mathrm{dcl}(\bar{b})$. Here, $\widetilde{f} = \widetilde{f}^\sigma$ should be understood as implying $\mathfrak{p} = \mathfrak{p}^\sigma$.

Remark. Since T is stable, any germ has a code.

Indeed, given a $\emptyset$-definable family f_z of partial functions, equality of germs at $\mathfrak{p}$, $\widetilde{f_b} = \widetilde{f_c}$, is defined by

$$E(b, c) := d_{\mathfrak{p}} x.(f_b(x) = f_c(x)),$$

and then since $\widetilde{f_b}^\sigma = \widetilde{f_{b^\sigma}}$, we have

$$\ulcorner \widetilde{f_b} \urcorner = \mathrm{dcl}(b/E).$$

Remark. Composition of germs is well-defined.

Say $\widetilde{f} : \mathfrak{p} \to \mathfrak{q}$ is **invertible** if it is invertible in the category of germs, i.e. $\widetilde{g} \circ \widetilde{f} = \widetilde{\mathrm{id}} = \widetilde{f} \circ \widetilde{g}$ for some $\widetilde{g} : \mathfrak{q} \to \mathfrak{p}$. Equivalently, $\widetilde{f}$ is injective on $\mathfrak{p}$.

Note that $\ulcorner \widetilde{f} \circ \widetilde{g} \urcorner \subseteq \mathrm{dcl}(\ulcorner \widetilde{f} \urcorner, \ulcorner \widetilde{g} \urcorner)$, and $\ulcorner \widetilde{f}^{-1} \urcorner = \ulcorner \widetilde{f} \urcorner$.

Definition. If p and q are stationary types and $\mathfrak{p}, \mathfrak{q}$ their global non-forking extensions, a **germ at** p is a germ at $\mathfrak{p}$, and $\widetilde{f} : p \to q$ means $\widetilde{f} : \mathfrak{p} \to \mathfrak{q}$.

Definition. Let $p, q, s \in S(\emptyset)$. A **family** $\widetilde{f_s}$ **of germs** $p \to q$ is the family $\widetilde{f_s} := (\widetilde{f_b})_{b \vDash s}$ of germs at p of a $\emptyset$-definable family f_z of partial functions, which is such that $\widetilde{f_b} : p \to q$ whenever $b \vDash s$.

The family is <u>canonical</u> if b is a code for $\widetilde{f_b}$, for all $b \vDash s$.

The family is <u>generically transitive</u> if $\widetilde{f_b}(x) \perp x$ for $(b, x) \vDash s \otimes p$.

Remark. $\widetilde{f_s}$ is generically transitive iff for $(x, y) \vDash p \otimes q$, there exists $b \vDash s|_x$ such that $\widetilde{f_b}(x) = y$.

Remark. Suppose $p, q, s \in S(\emptyset)$, and $\widetilde{f_s}$ is a family of germs $p \to q$. Let $b \vDash s$ and $x \vDash p|_b$, and let $y = \widetilde{f_b}(x)$. Then

$$x \perp b;\ y \perp b;\ y \in \mathrm{dcl}(bx). \tag{1}$$

Conversely, if (b, x, y) satisfy (1), let $f_b(x) = y$ be a formula witnessing $y \in \mathrm{dcl}(bx)$. Then $\widetilde{f_s} := (\widetilde{f_b})_{b \vDash s}$ is a family of germs $p \to q$, where $s = \mathrm{tp}(b)$, $p = \mathrm{tp}(x)$, $q = \mathrm{tp}(y)$.

Lemma 3.1. *In the correspondence of the previous remark,*

(i) $\widetilde{f_b}$ *can be chosen to be invertible iff also* $x \in \mathrm{dcl}(by)$*;*

(ii) the family $\widetilde{f_s}$ *is generically transitive iff* $x \perp y$*;*

(iii) $\ulcorner \widetilde{f_b} \urcorner = \mathrm{Cb}(xy/b)$ *(so* $\widetilde{f_s}$ *is canonical iff* $\mathrm{Cb}(xy/b) = b$*).*

Proof.

(i) $x \in \mathrm{dcl}(by)$ iff f_b can be taken to be injective at x.

(ii) Immediate.

(iii) Let $\sigma \in \mathrm{Aut}(\mathbb{M})$. Let $\mathfrak{p}$ be the global nonforking extension of p, so $\mathfrak{p}^\sigma = \mathfrak{p}$. Let $\mathfrak{r}$ be the global nonforking extension of $\mathrm{stp}(xy/b)$. Then $\mathfrak{r}$ is equivalent to $\mathfrak{p}(x) \cup \{y = f_b(x)\}$.

$$\begin{aligned} \mathrm{Cb}(xy/b)^\sigma = \mathrm{Cb}(xy/b) &\Leftrightarrow \mathrm{Cb}(\mathfrak{r})^\sigma = \mathrm{Cb}(\mathfrak{r}) \\ &\Leftrightarrow \mathfrak{r}^\sigma = \mathfrak{r} \\ &\Leftrightarrow \mathfrak{p}(x) \cup \{y = f_{b^\sigma}(x)\} \equiv \mathfrak{p}(x) \cup \{y = f_b(x)\} \\ &\Leftrightarrow \mathfrak{p}(x) \vDash f_{b^\sigma}(x) = f_b(x) \\ &\Leftrightarrow \widetilde{f}_b^\sigma = \widetilde{f}_b. \end{aligned}$$

□

4 Homogeneous spaces

Let (G, X) be a $\bigwedge$-definable homogeneous space over $\emptyset$, i.e. G is a $\bigwedge$-definable group and X is a $\bigwedge$-definable set, and $* : G \times X \to X$ is a relatively definable transitive group action, all over $\emptyset$.

Say (G, X) is **connected** if G is connected, i.e. G has no relatively definable proper subgroup of finite index.

Say (G, X) is **faithful** if the action is faithful, i.e. only the identity element of G acts trivially on X.

Recall that a complete type extending G is generic if it is generic for the left, equiv right, multiplication action of G on itself.

Fact 4.1. *(i) G is connected iff it has for any A a unique generic type over A, iff it has a unique generic type over $\emptyset$.*

(ii) For $b \in X$, $\mathrm{tp}(b/A)$ is generic iff $\forall g \in G.(b \mathop{\smile\!\!\!\!\!\!|}_A g \Rightarrow g * b \mathop{\smile\!\!\!\!\!\!|}_A g)$.

(iii) G acts transitively on the set of global generic types of X.

Lemma 4.2. *Suppose $p \in S(\emptyset)$ extends X and $\forall g \in G.(b \vDash p|_g \Rightarrow g * b \vDash p|_g)$. Then p is the unique generic of X over $\emptyset$.*

In particular, if this holds for the left/right multiplication action of G on itself, then G is connected.

Proof. p is generic by Fact 4.1(ii).

If $p' \in S(\emptyset)$ is generic, by Fact 4.1(iii) there is g such that $g * \mathfrak{p} = \mathfrak{p}'$, where $\mathfrak{p}$ and $\mathfrak{p}'$ are the global non-forking extensions. So if $b \vDash p|_g$ then $g * b \vDash p'$, but also $g * b \vDash p$, so $p' = p$. □

5 Hrushovski-Weil

Definition. Let $p, s \in S(\emptyset)$. A family $\widetilde{f_s}$ of germs $p \to p$ is

- **closed under inverse** if for $b \models s$ there exists $b' \models s$ such that $\widetilde{f_b}^{-1} = \widetilde{f_{b'}}$;
- **closed under generic composition** if for $b_1 b_2 \models s^{(2)}$, there exists b_3 such that
$$\widetilde{f_{b_1}} \circ \widetilde{f_{b_2}} = \widetilde{f_{b_3}}$$
and $b_i b_3 \models s^{(2)}$ for $i = 1, 2$.

Theorem 5.1 (Hrushovski-Weil for actions)**.** *Let $p, s \in S(\emptyset)$, and suppose $\widetilde{f_s}$ is a generically transitive canonical family of invertible germs $p \to p$ which is closed under inverse and generic composition.*

Then there exists a connected faithful $\bigwedge$-definable/$\emptyset$ homogeneous space (G, X), a definable embedding (i.e. relatively definable injection) of s into G as its unique generic type, and a definable embedding of p into X as its unique generic type, such that the generic action of s on p agrees with that of G on X, i.e. $\widetilde{f_b}(a) = b{*}a$ for $(a, b) \models p \otimes s$.

This is related to the following more intuitive result:

Theorem (Hrushovski-Weil group chunk theorem)**.** *Let $p \in S(\emptyset)$, and suppose $*$ is a definable partial binary function such that:*

- *If $(a, b) \models p^{(2)}$ then $a * b$ is defined, and $(a, a * b) \models p^{(2)}$ and $(b, a * b) \models p^{(2)}$;*
- *If $(a, b, c) \models p^{(3)}$ then $(a * b) * c = a * (b * c)$.*

Then one obtains a $\bigwedge$-definable connected group G and a definable embedding of p into G such that if $(a, b) \models p^{(2)}$ then $a * b$ agrees with the product in G.

(In the case T = ACF (where $\bigwedge$-definable groups are actually algebraic groups (up to definable isomorphism)), this is essentially a theorem of Weil.)

Note that this is not quite just a matter of applying Theorem 5.1 with $f_b(a) := b * a$ – we didn't assume closedness under inverse nor canonicity, so some extra argument is involved.

Proof of Theorem 5.1. Let G be the group of germs generated by $\widetilde{f_s}$.

Claim 5.2. *Any element of G is a composition of two generators.*

Proof. Since the family is closed under inverses, the identity is the composition of two generators.

By generic composability and completeness of s, any generator $\widetilde{f_b}$ is the composition of two generators.

So it suffices to see that any composition of three generators
$$\widetilde{f_{b_1}} \circ \widetilde{f_{b_2}} \circ \widetilde{f_{b_3}}$$
is the composition of two.

Let $b' \models s|_{b_1 b_2 b_3}$. Then
$$\widetilde{f_{b_1}} \circ \widetilde{f_{b_2}} \circ \widetilde{f_{b_3}} = \widetilde{f_{b_1}} \circ \widetilde{f_{b'}} \circ \widetilde{f_{b'}}^{-1} \circ \widetilde{f_{b_2}} \circ \widetilde{f_{b_3}}$$

Now $b' \mathop{\smile\!\!\!\!\!\mid} b_2$, so say $\widetilde{f}_{b'}^{-1} \circ \widetilde{f}_{b_2} = \widetilde{f}_{b''}$ with $b'' \vDash s$ independent from b' and from b_2.

Now $b' \mathop{\smile\!\!\!\!\!\mid}_{b_2} b_3$, so $b'' \mathop{\smile\!\!\!\!\!\mid}_{b_2} b_3$, since $b'' \in \operatorname{dcl}(b'b_2)$, so since $b'' \mathop{\smile\!\!\!\!\!\mid} b_2$, we have $b'' \mathop{\smile\!\!\!\!\!\mid} b_3$.

Also $b' \mathop{\smile\!\!\!\!\!\mid} b_1$.

So the germs $\widetilde{f}_{b_1} \circ \widetilde{f}_{b'}$ and $\widetilde{f}_{b'}^{-1} \circ \widetilde{f}_{b_2} \circ \widetilde{f}_{b_3}$ appear in the family $\widetilde{f}_s$. □

So G is $\bigwedge$-definable as pairs of realisations of s modulo equality of the corresponding composition of germs, and the group operation is relatively definable as composition of germs. We identify s with its image in G under the definable embedding $b \mapsto \widetilde{f}_b$.

We show that G is connected with generic type s via Lemma 4.2. We show that if $g \in G$ and $b \vDash s|_g$, then $g \cdot b \vDash s|_g$. This holds for $g \vDash s$, by closedness under generic composition. Let $g \in G$, say $g = g_1 \cdot g_2$ with $g_1, g_2 \vDash s$. Let $b \vDash s|_{g_1,g_2}$. Then $g_2 \cdot b \vDash s|_{g_1,g_2}$, and so $g_1 \cdot g_2 \cdot b \vDash s|_{g_1,g_2}$. Now $g = g_1 \cdot g_2 \in \operatorname{dcl}(g_1, g_2)$, so $b \vDash s|_g$ and $g \cdot b \vDash s|_g$, as required.

G acts generically on p by application of germs; set $g * a := g(a)$ if $a \vDash p|_g$.

Now define $X := (G \times p)/E$ where $(g, a)E(g', a')$ iff $(h \cdot g) * a = (h \cdot g') * a'$ for $h \vDash s|_{aa'gg'}$, which is definable by definability of s.

Define the action of G on X by $h * ((g, a)/E) := (h \cdot g, a)/E$. This is well-defined, since if $(g, a)E(g', a')$ and $h \in G$, then if $h' \vDash s|_{g,g',a,a',h}$, then also $h' \cdot h \vDash s|_{g,g',a,a',h}$ by genericity, and we have $(h' \cdot h \cdot g) * a = (h' \cdot h \cdot g') * a'$, so $(h \cdot g, a)E(h \cdot g', a')$. p definably embeds into X via $a \mapsto (1, a)/E$.

We show transitivity. Let $a, a' \vDash p$, and we show $(1, a')/E \in G*(1, a)/E$; this suffices for transitivity, since clearly $(G, a')/E \subseteq G * (1, a')/E$. Let $c \vDash p|_{aa'}$. Then by generic transitivity of $\widetilde{f}_s$, there exist $g \vDash s|_a$ and $g' \vDash s|_c$ such that $g * a = c$ and $g' * c = a'$. Then $(h \cdot g) * a = h * c$ for $h \vDash s|_{acg}$, so $(g, a)E(1, c)$. Similarly $(g', c)E(1, a')$. So $(g' \cdot g) * (1, a)/E = g' * (g, a)/E = g' * (1, c)/E = (g', c)/E = (1, a')/E$.

For faithfulness of the action: suppose g acts trivially, and let $a \vDash p|_g$. Then $(g, a)E(1, a)$, so let $h \vDash s|_{ag}$; then $(h \cdot g) * a = h * a$. But $h \mathop{\smile\!\!\!\!\!\mid}_g a$, hence also $h \cdot g \mathop{\smile\!\!\!\!\!\mid}_g a$, so $h, h \cdot g \mathop{\smile\!\!\!\!\!\mid} a$ since $g \mathop{\smile\!\!\!\!\!\mid} a$. So $h \cdot g = h$ as germs, so $g = 1$.

Finally, we conclude from Lemma 4.2 that p is the unique generic type, since if $g \in G$ and $a \vDash p|_g$, then $g * a \vDash p|_g$ as this is the action of a germ. □