Pseudofiniteness, unimodularity, and planar incidence patterns

Motivation and speculations

Theorem [Green-Tao]:
For $n >> 0$,
given n points in $\mathbb{R}^2$,
there are at least $n/2$ straight lines passing through exactly 2 points ("ordinary lines").

Recall:
Let $E \subseteq \mathbb{R}^2$ be an elliptic curve, WLOG (i.e. after projective transformation) in Weierstrass form;
$E(\mathbb{R})$ has an abelian group structure given by:
$0 =$ point at infinity;
$x + y + z = 0$ iff x, y, z are the intersections with a straight line (counting multiplicity).

Theorem [Green-Tao]:
Exists C s.t.
given $K > 0$,
for $n >> 0$,
given n points in $\mathbb{R}^2$,
suppose there are $< Kn$ ordinary lines,
and suppose there are $< C$ points on each line,
then up to CK points,
the n points form a coset $H + a$ in the (abelian) group of smooth points of an irreducible cubic plane curve,
with $3a \in H$.

Fact:
$H \ = \mathbb{Z}/N\mathbb{Z}$ or $H \ = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

Non-standard formulation:
Let P be a pseudofinite subset of $\mathbb{R}^2$, equipped with the ternary relation I
$I(x, y, z)$ iff x, y, z distinct and co-linear.

Suppose $|I(a, b, P)| < \infty$,
and $|O(a, P)| < \infty$ where $O(x, y) \equiv \neg\exists z.I(x, y, z)$.

Then there is $P' \subseteq P$ cofinite (definable??) which is a cofinite subset of a coset $H + a$ of a pseudofinite subgroup H of the smooth points of an irreducible cubic plane curve (over $*\mathbb{R}$),
with $3a \in H$,
and $I(x, y, z) \leftrightarrow x + y + z = 0$ for distinct $x, y, z \in P'$.

Furthermore, H is pseudocyclic or $H \ = \mathbb{Z}/2\mathbb{Z} \times C$ for C pseudocyclic.

Recall:
A type-definable set D is **minimal** if for any B (with D defined over B), there is a unique non-algebraic $p_B \in S(B)$ on D.

p_B is stationary, U-rank 1.
acl is a pregeometry on a minimal set.
Buechler's dichotomy:
if the pregeometry is not locally modular,
then D is strongly minimal (i.e. a type-definable subset of a strongly minimal definable set).

Remark:
$(P'; I)$ is bi-interpretable with $(H; +)$,
which is superstable unidimensional;
the divisible elements form a minimal set,
which is locally modular, non-trivial.
(More generally: have a map $H \to \widehat{\mathbb{Z}} \times \mathbb{Z}/2\mathbb{Z}$,
namely the inverse limit of $[n]$,
and the fibres are minimal.)

Proof:
Note that $(h + a) + (h' + a) + (h'' + a) = 0 \leftrightarrow h + h' + h'' = 0$.
So $x - x' = y - y'$ iff $\exists z. z + x + y' = 0 = z + x' + y$,
so this interprets H,
with addition given by $(x - x') + (x' - x'') = (x - x'')$.
A pseudocyclic group G, quotients of torsion subgroups $G[n]/G[m]$ are finite, and so are quotients of divisibility subgroups nG/mG.
These are the pp-definable subgroups ([Prest] 2.$\mathbb{Z}$1). So there are no infinite descending chains of such with infinite quotients, so by [Prest] 3.1,
G is superstable.
Minimality:
by this quantifier elimination, 1-types are determined by the cosets of these subgroups they lie in; we've explicitly decided on the divisibility subgroups, so only new thing that can happen over parameters is $pg = b$,
but then g is algebraic over b since $G[p]$ is finite.
Unidimensionality:
these minimal types are manifestly non-orthogonal, because we can shift between them by adding appropriate parameters.

Question [Green-Tao]:
Suppose a positive proportion of lines have > 2 points,
and a line has boundedly many points.
Does it follow that a positive proportion of the points lie on a cubic?

Hope:
A model-theoretic proof of the theorem should exist,
and maybe also a positive answer to the question.
Magicking group structures out of the vaguest traces is something model theorists have been doing for decades;
moreover, this is in the end a minimal locally modular non-trivial structure,
and it is classical that any such looks a bit like this one,
and in particular it's relatively easy to find the group.

Moreover, the local modularity follows from minimality and pseudofiniteness, via unimodularity.

So even though there's no obvious direct proof of minimality,
the group construction techniques may apply.

Preliminaries

Definition:
A minimal set D is **unimodular** if for any generic k_1-to-k_2 correspondence on D^n,
meaning a definable $X \subseteq D^n \times D^n$ with $\mathrm{RM}(\pi_i(X)) = n$ and $|\pi_i^{-1}(x)| = k_i$,
$k_1 = k_2$.

Equivalently: if $a, b \models p^{(n)}$, then $\mathrm{mult}(a/b) = \mathrm{mult}(b/a)$.

Examples:
Algebraically closed fields are not unimodular;
consider $X = \{(x, y) \mid y = x^2\}$ (if char $\neq 2$).

Modules over division rings are unimodular.

So is e.g. $(\mathbb{C}; \{x^2 + y^2 + z^2 = 0\})$.

Strongly minimal pseudofinite $\to$ unimodular

Following Pillay's note "Strongly minimal pseudofinite structures".

Let $M = \Pi_{\mathcal{U}} M_i$ be pseudofinite.

Let D be a strongly minimal set in M.

Lemma:
Let $X \subseteq D^n$ be definable.
There exists $p_X(q) \in \mathbb{Z}[q]$ such that almost always

$$|X| = p_X(|D|)$$

Proof:
By induction on Morley rank and degree of X.
Let $X' \subseteq X$ with $\mathrm{RM}(X') = \mathrm{RM}(X) =: n$, $\deg(X) = 1$,
and a projection map $\pi : X' \to D^n$ with all fibres of size k and $\mathrm{RM}(\pi(X')) = n$.

By induction, we have polynomials for $X \setminus X'$ and $D^n \setminus \pi(X')$.
So set

$$p_X(q) := p_{X \setminus X'}(q) + k(q^n - p_{D^n \setminus \pi(X')}(q)).$$

$\square$

Proposition:
D is unimodular.

Proof:
Suppose X is a generic k_1-to-k_2 correspondence,
$D^n \leftarrow^{k_1} X \rightarrow^{k_2} D^n$.

Then $p_X(q) = k_i p_{\pi_i(X)}(q)$,
but $p_{\pi_i(X)}(q)$ has leading term q^n.
□

Local modularity

Theorem [Hrushovski-Zilber]:
A unimodular minimal type is locally modular.

Theorem [Pillay]:
Any minimal set in a pseudofinite theory is locally modular.

Hence, a pseudofinite theory of finite U-rank is 1-based,
i.e. $A \mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits_{acl^{eq}(A)\cap \mathrm{acl}^{eq}(B)} B$.

Proof:
By the Buechler dichotomy, any U-rank 1 type is either locally modular or strongly minimal.
So by pseudofiniteness, every U-rank 1 type is locally modular.
1-basedness follows by standard results (also due to Buechler).

Theorem:
If D is non-trivial locally modular minimal,
there is a type-definable minimal, hence abelian, group G,
and D is in definable generic finite-to-finite correspondence with G.

Moreover, G is 1-based, so the relatively definable subsets of G^n are boolean combinations of cosets of $\mathrm{acl}(\emptyset)$-definable subgroups.

Remark:
pseudofinite + stable =/> 1-based (MacPherson-Tent).

pseudofinite + SU-rank 1 =/> 1-based (pseudofinite fields).

Proof sketch of unimod → loc.mod:
Details are in GST (section 2.4 up to 2.4.15, and section 5.3).

Define **Zilber degree** $Z(a) := \mathrm{mult}(a/d)$ where $d \models p^{(n)}$ is interalgebraic with a.
Unimodularity → well-defined.
$Z(a/b) := Z(ab)/Z(b)$.
If X is a Morley degree 1 partial type over b,
$Z(X) := Z(a/b)$ where $a \in X$ is generic over b.

Show that if $X_b, Y_c \subseteq D^2$ are $\geq$ 2-dimensional families of minimal sets ("plane curves"), then for generic (b, c), $|X_b \cap Y_c| = Z(X_b)Z(Y_c)$.

(Proof: $b \mathop{\smile\!\!\!\!\!\!\!\;|} c$; using the rank condition, one sees that WLOG also $b \mathop{\smile\!\!\!\!\!\!\!\;|}_a c$ with $a \in X_b \cap Y_c$; then

$$Z(a/bc) = Z(abc)/Z(bc) = Z(c/ab)Z(a/b)Z(b)/Z(bc) = Z(a/c)Z(a/b)Z(c)Z(b)/Z(bc) = Z(a/b)Z(a/c)$$

where we used

$$Z(a/c)Z(c) = Z(ac) = Z(c/a)Z(a) = Z(c/a) = Z(c/ab)$$

since $Z(a) = 1$ and $b \mathop{\smile\!\!\!\!\!\!\!\;|}_a c$,
and

$$Z(bc) = Z(b)Z(c)$$

since $b \downarrow c$.)

Conclude **2-pseudolinearity** - there are no 3-dimensional families of plane curves.

Indeed,
if X_a is such,
take $a' \models tp(a)|_a$,
let $c \in X_a \cap X_{a'}$;
let X'_a be the non-forking extension to ac,
which still has dim ≥ 2, so

$$|X'_a \cap X'_{a'}| = Z(X'_a)Z(X'_{a'}) = Z(X'_a)^2 = Z(X_a)^2 = |X_a \cap X_{a'}|,$$

which is nonsense.

Finally,
suppose there is a 2-dimensional family of plane curves;
consider such as a correspondence on D,
by considering compositions, using that the dimension doesn't go up when we do so, and using usual tricks (skipping lots, see 5.3.3 and paragraph after it),
get a dim 2 definable group action,
which by stable group theory yields a definable field,
contradicting pseudolinearity (or, more directly, unimodularity).

So D is linear, i.e. locally modular.