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Definability in the infinitesimal subgroup of a compact Lie group

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Let $(G;*)$ be a simple centreless compact linear algebraic group $/\mathbb{R}$ (// compact Lie)

e.g. $G = SO_3$

I: $(G(\mathbb{R});*)$

Thm [Nesin-Pillay '91]: $X \subseteq G(\mathbb{R})^n$ is $(G(\mathbb{R});*)$-def^{ble} iff semialgebraic (i.e. $(\mathbb{R};+,\cdot)$-def^{ble})

(Moreover, $(G(\mathbb{R});*)$ is bi-interpretable with $(\mathbb{R};+,\cdot)$)

pf rem^{ks}: $\Rightarrow$: clear

$\Leftarrow$: (i) Interpret copy of $(SO_3(\mathbb{R});*)$ in $(G(\mathbb{R});*)$

(ii) In $SO_3(\mathbb{R})$, $\{\alpha : \alpha^2 = e\}$ ~ points

$\{l_\alpha := \{\beta : \beta^2 = e,\ \beta \neq \alpha,\ [\alpha,\beta] = e\} : \alpha^2 = e\}$ ~ lines } of an $\mathbb{RP}^2$

(iii) hence get $(G(\mathbb{R});*)$-def^{ble} field $(K;+,\cdot) \cong (\mathbb{R};+,\cdot)$ ⇝ (iv) $G(\mathbb{R}) \cong G(K)$ $(G(\mathbb{R});*)$-def^{ble} □

Thm [Peterzil-Pillay-Starchenko '00]: A def^{bly} simple group (abelian, no $(G;*)$-def^{ble} proper normal subgrp) def^{ble} in an o-min^l str^{re} is biinterpretable with an RCF or an ACF (and def^{ble} sets are semialg resp Zariski constructible)

II: $(G^{00};*)$

Let $R \succ \mathbb{R}$ e.g. $R = \mathbb{R}^{\mathcal{U}}$ (← non-principle u-filter on ω) or $R = \mathbb{R}\{\{t\}\}$

$st : R \dashrightarrow \mathbb{R}$ $\quad \mathcal{O} := \mathrm{dom}\, st$; $m := st^{-1}(0)$ $(= \bigcap_{n\in\mathbb{N}} [-\frac{1}{n}, \frac{1}{n}])$

induces $st_G : G(\mathcal{O}) \twoheadrightarrow G(\mathbb{R})$

$G^{00}(R) = \ker(st_G) = st_G^{-1}(e) = (I + \mathrm{Mat}_n(m)) \cap G(R)$

G^{00} = smallest $\bigwedge$-def^{ble} subgroup of bounded index ← Pillay

$1 \to G^{00}(R) \to G(\mathcal{O}) \xrightarrow{st_G} G(\mathbb{R}) \to 1$

Thm [B-Peterzil]:

$X \subseteq G^{00}(R)^n$ is $(G^{00}(R);*)$-def^{ble} iff $(R;+,\cdot,\mathcal{O})$-def^{ble}

(moreover, $(G^{00}(R);*)$ bi-terp^{ble} with $(R;+,\cdot,\mathcal{O})$)

Rem: $Th(R;+,\cdot,\mathcal{O})$ = RCVF = Th(non-trivially valued real closed field)

- QE in $(+,\cdot,<,v(x)\leq v(y),0,1)$
- weakly o-minimal

Example: $\{(A,B) \in G^{00} : \det(A-B) = 0\}$ is $(G^{00};*)$-def^{ble} (as matrices)

WLOG $R = \mathbb{R}^{\mathcal{U}}$, so $\aleph_1$-sat^{d}.

First, suppose $G = SO_3$.

III Defining a field in $(SO_3^{00}(R);*)$

- A group interval $I = [-p,p]$ is an interval in a convex $\bigwedge$-def^{ble} ordered group, subset of an o-min^l str^{re}

Thm [Peterzil-Starchenko '97]:

If J is a group interval in an $\aleph_1$-sat^d o-min^l structure M, $M_J :=$ [J with induced structure],

then either • M_J is a reduct of V_I, V an ordered vector space over an ordered div^n ring, $I \subseteq (V;+)$ group interval

or • an RCF is def^{ble} in M_J on a subinterval $K \subseteq J$ (and the orders agree).

Now $S := SO_3(R)$, $S^{00} := SO_3^{00}(R)$

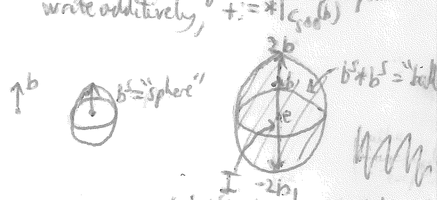
Let $b \in S^{00} \setminus \{e\}$.

$C_S(b)$ = {rotations with same axis as b} $\cong SO_2(R)$

$C_{S^{00}}(b) = C_S(b) \cap S^{00} \cong SO_2^{00}(R)$ is a convex $\bigwedge$-def^{ble} ordered group. write additively, $+ := *|_{C_{S^{00}}(b)}$

$I := b^S * b^S \cap C_{S^{00}}(b)$

$= [-2b, 2b]$ in $C_{S^{00}}(b)$



($b^S * b^S$ is closed, connected, S-invariant)

// but want gr^p int^l def^{ble} in $(S^{00};*)$

$X := b^{S^{00}} * b^{S^{00}} \cap C_{S^{00}}(b) \subseteq I$ $\quad (S^{00};*)$-def^{ble}

$X \supseteq (h, 2b]$ some $h < 2b$

$p := 2b - h$; then $J := (-p,p) \subseteq C_{S^{00}}(b)$ group int^l

and $(J;+,<)$ is $(S^{00};*)$-def^{ble}

(since $(0,p) = (X-h) \cap (2b-X)$ is)

Now let $h_1, h_2 \in S^{00}$ s.t.

$\phi : J^3 \to S^{00}$

$\phi(x,y,z) = x * y^{h_1} * z^{h_2}$

is non-singular at $(0,0,0)$ (holds if $[h_i, b] \neq e \neq [h_1,h_2]$)

so shrinking J, ϕ is a bij^n with a nbd of e in G^{00}.

Define $** : J^3 \times J^3 \dashrightarrow J^3$ by $x ** y = z \Leftrightarrow \phi(x)*\phi(y) = \phi(z)$

then $(J;+,<,**)$ is $(S^{00};*)$-def^{ble} and $(R;+,\cdot)$-def^{ble}

not a reduct of a gr^p int^l in an ordered v.s.

so P-S $\Rightarrow$ field $(K;+,\cdot)$ on subint^l is $(S^{00};*)$-def^{ble} and $(R;+,\cdot)$-def^{ble}

§4 $S^{00} \hookrightarrow GL_3(K)$

Otero-Peterzil-Pillay ⇝ $\theta_R : (R;+,\cdot) \xrightarrow{\cong} (K;+,\cdot)$ ($(R;+,\cdot)$-def^{ble}). Induces $\theta_S : S(R) \xrightarrow{\cong} S(K)$

$\phi : K^3 \to S^{00}$

Differentiating in K ⇝ $(R;+,\cdot)$-def^{ble} embedding $\mathrm{Ad} : S \hookrightarrow GL_3(K)$

$g \mapsto (D_e(x^g))$

$\mathrm{Ad}|_{S^{00}} : S^{00} \hookrightarrow GL_3(K)$ is $(S^{00};*)$-def^{ble}

$\mathrm{Ad} \circ \theta_S^{-1} : S(K) \hookrightarrow GL_3(K)$ is $(R;+,\cdot)$-def^{ble} hence $(K;+,\cdot)$-def^{ble} hence $(S^{00};*)$-def^{ble}

so $\theta_S|_{S^{00}} = (\mathrm{Ad} \circ \theta_S^{-1})|_{S^{00}}^{-1} \circ \mathrm{Ad}|_{S^{00}}$ is $(S^{00};*)$-def^{ble}

$: S^{00}(R) \xrightarrow{\cong} S^{00}(K)$

so also $S^{00}(K)$ is $(S^{00};*)$-def^{ble}.

So if $X \subseteq S^{00}(R)^n$ is $(R;+,\cdot,\mathcal{O})$-def^{ble}, hence $(R;+,\cdot,S^{00}(R))$-def^{ble}

then $\theta_S(X) \subseteq S^{00}(K)^n$ is $(K;+,\cdot,S^{00}(K))$-def^{ble}

hence $(S^{00};*)$-def^{ble},

so $X = \theta_S|_{S^{00}}^{-1}(\theta_S(X))$ is $(S^{00};*)$-def^{ble}

so done in the case $G = SO_3$.

§5 Defining an SO_3^{00} in G^{00}

G simple compact $/\mathbb{R}$

Lemma: Exists $(\mathbb{R};+,\cdot)$-def^{ble} closed Lie subgroup $S(\mathbb{R}) \leq G(\mathbb{R})$ isomorphic to $SO(3)$ or $Spin(3)$

s.t. $S^{00}(R)$ is $(G^{00};*)$-def^{ble}.

Then proceed as above.

Pf of Lemma: Structure of compact Lie algebras

⇝ exists subalgebra $\mathfrak{s}' \leq \mathfrak{g} := L(G)$

s.t. $\mathfrak{s}' = C_{\mathfrak{g}}(C_{\mathfrak{g}}(\mathfrak{s}'))$ and $\mathfrak{s} := [\mathfrak{s}', \mathfrak{s}'] \cong \mathfrak{so}_3$

$\mathfrak{s}' = L(S')$ $\mathfrak{s} = L(S)$

Namely: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} L_\alpha$ ($\alpha \in (\mathfrak{h}\otimes\mathbb{C})^*$, $i\alpha \in \mathfrak{h}^*$)

$\mathfrak{h}$ Cartan $\dim L_\alpha = 2$

$L_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}$ ↑ root space in $\mathfrak{g}\otimes\mathbb{C}$

$\alpha_0 \in \Delta$

$\mathfrak{s}$:= subalgebra gen^d by L_{α_0} $= L_{\alpha_0} \oplus i[\mathfrak{g}_{\alpha_0}, \mathfrak{g}_{-\alpha_0}]$

$\mathfrak{s}' := \mathfrak{h} \oplus L_{\alpha_0}$

$C_{\mathfrak{g}}(\mathfrak{s}') = \ker(\alpha_0) \cap \ldots$

def^{ble} in $(\mathbb{R};+,\cdot)$ resp $(G^{00};*)$ since centralisers of finite by Noetherianity in RCF

$\vdash C_G(C_G(S')) = S'$; $[S', S']_1 = S$ $\Rightarrow$ S is $(\mathbb{R};+,\cdot)$-def^{ble}

$\vdash C_{G^{00}}(C_G(S')^{00}) = S'^{00}$; $[S'^{00}, S'^{00}]_1 = S^{00}$ $\Rightarrow$ S^{00} is $(G^{00};*)$-def^{ble} □