1 Setup

1.1 Pseudofinite dimension

- $\mathcal{U} \subseteq \mathbb{P}(\omega)$ non-principal ultrafilter.
- $K := \mathbb{C}^{\mathcal{U}}$.
- $X \subseteq K^n$ is **internal** if $X = \prod_{s \to \mathcal{U}} X_s$ for some $X_s \subseteq \mathbb{C}^n$.
- Then set $|X| := \prod_{s \to \mathcal{U}} |X_s| \in \mathbb{N}^{\mathcal{U}} \cup \{\infty\}$.
- Fix $\xi \in \mathbb{N}^{\mathcal{U}}$ with $\xi > \mathbb{N}$.

  Definition 1.1 (Coarse pseudofinite dimension $\boldsymbol{\delta}$)**.** For X internal,

$$\boldsymbol{\delta}(X) = \boldsymbol{\delta}_\xi(X) := \operatorname{st}\left(\frac{\log(|X|)}{\log(\xi)}\right) \in \mathbb{R}_{\geq 0} \cup \{-\infty, \infty\}.$$

- *Note that internality is closed under cardinality quantifiers:* If $R \subseteq K^n \times K^m$ is internal and $\alpha \in \mathbb{R}^{\mathcal{U}}$, then $\{\overline{y} \in K^n : \exists_{\geq \alpha} \overline{x}. R(\overline{x}, \overline{y})\}$ is internal.

1.2 $\mathcal{L}_{\text{int}}$ monster

- $\mathcal{L}_{\text{int}}$: predicate for each internal $X \subseteq K^n$.
- $\mathbb{K} \succ K$ monster model in $\mathcal{L}_{\text{int}}$.
- For $\phi \in \mathcal{L}_{\text{int}}$, set $\boldsymbol{\delta}(\phi) := \boldsymbol{\delta}(\phi(K))$.
- $\boldsymbol{\delta}$ has a unique extension to $(\mathcal{L}_{\text{int}})_{\mathbb{K}}$ such that

$$\begin{aligned} \operatorname{tp}(\overline{b}) &\mapsto \boldsymbol{\delta}(\phi(\overline{x}, \overline{b})) \\ S_{\overline{y}}(\emptyset) &\to \{-\infty\} \cup \mathbb{R} \cup \{\infty\} \end{aligned}$$

  is well-defined and continuous for each $\phi(\overline{x}, \overline{y}) \in \mathcal{L}_{\text{int}}$.

- Explicitly, $\boldsymbol{\delta}(\phi(\overline{x}, \overline{a})) := \sup\{q \in \mathbb{Q} : \mathbb{K} \vDash \exists_{\geq \xi^q} \overline{x}.\ \phi(\overline{x}, \overline{a})\}$.
- For Φ a partial type, $\boldsymbol{\delta}(\Phi) := \inf\{\boldsymbol{\delta}(\phi) : \Phi \vDash \phi\}$.
- $\boldsymbol{\delta}(a/C) := \boldsymbol{\delta}(\operatorname{tp}(a/C))$.

  Fact 1.2. *For $C \subseteq \mathbb{K}$ small and $a, b \in \mathbb{K}^{<\omega}$,*

  (i) $a \equiv_C b \Rightarrow \boldsymbol{\delta}(a/C) = \boldsymbol{\delta}(b/C)$.

  (ii) $\boldsymbol{\delta}(ab/C) = \boldsymbol{\delta}(a/bC) + \boldsymbol{\delta}(b/C)$.

  (iii) A partial type Φ over C has a realisation $a \in \Phi(\mathbb{K})$ with $\boldsymbol{\delta}(a/C) = \boldsymbol{\delta}(\Phi)$.

1.3 acl^0

We have $\mathbb{C} \leq \mathbb{C}^{\mathcal{U}} \leq \mathbb{K}$.

Definition 1.3. *Superscript 0 means: reduct to* $\mathrm{ACF}_{\mathbb{C}}$.
Work in $\mathbb{K}^{\mathrm{eq}0} := \{\mathrm{ACF}-\text{imaginaries}\}$ ("$= \mathbb{K}^{<\omega}$ ").

- $d^0(B) := \mathrm{trd}(B/\mathbb{C})$
- $a \in \mathrm{acl}^0(B)$ iff $d^0(a/B) = \mathrm{trd}(a/\mathbb{C}(B)) = 0$.

Remark 1.4. $a \in \mathrm{acl}^0(B) \Rightarrow \boldsymbol{\delta}(a/B) = 0$.

2 Coherence

Definition 2.1. $P \subseteq \mathbb{K}$ is **coherent** if for any tuple $\overline{a} \in P^{<\omega}$,

$$\boldsymbol{\delta}(\overline{a}) = d^0(\overline{a}).$$

In other words, $\boldsymbol{\delta}$ *is equal on* $P^{<\omega}$ *to the dimension function of the pregeometry* $(P; \mathrm{acl}^0)$.
More generally:

Definition 2.2. $a \in \mathbb{K}^{\mathrm{eq}0}$ is in **coarse general position** (or is **cgp**) if for any $B \subseteq \mathbb{K}$,

$$d^0(a/B) < d^0(a) \Rightarrow \boldsymbol{\delta}(a/B) = 0.$$

Any $a \in \mathbb{K}$ is cgp.

Definition 2.3. $P \subseteq \mathbb{K}^{\mathrm{eq}0}$ is **coherent** if

- every $a \in P$ is cgp, and
- for any tuple $\overline{a} \in P^{<\omega}$,
$$d^0(\overline{a}) = \boldsymbol{\delta}(\overline{a}).$$

Then

- $(P; \mathrm{acl}^0)$ is a pregeometry,
- if $d^0(a) = k$ $(\forall a \in P)$, then $\boldsymbol{\delta}(\overline{a}) = k \dim_P(\overline{a})$ for any $\overline{a} \in P^{<\infty}$.

2.1 Modularity of coherence

Definition 2.4. For $P \subseteq \mathbb{K}^{\mathrm{eq}0}$,

$$\mathrm{ccl}(P) := \{x \in \mathrm{acl}^0(P) : \{x\} \text{ is coherent}\}.$$

Lemma 2.5. *If* P *is coherent, so is* $\mathrm{ccl}(P)$.

Proposition 2.6. *Suppose* $P = \mathrm{ccl}(P)$ *is coherent.*
Then $(P; \mathrm{acl}^0)$ *is a modular pregeometry.*
(i.e. for $a, b \in P$ *and* $C \subseteq P$*, if* $a \in \mathrm{acl}^0(bC) \setminus (\mathrm{acl}^0(C) \cup \mathrm{acl}^0(b))$ *then exists* $c \in \mathrm{acl}^0(C)$ *such that* $a \in \mathrm{acl}^0(bc)$*.)*

(Eventual) consequence of Szemeredi-Trotter theorems

Example 2.7. Suppose G is a complex abelian algebraic group, $F \leq \mathrm{End}^0(G) := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(G)$ a division ring, $A \subseteq G(\mathbb{K})$ independent generics.

Let $Q := \{\Sigma_i \eta_i(a_i) : \eta_i \in F \cap \mathrm{End}(G); a_i \in A\} \subseteq \mathbb{K}^{\mathrm{eq}0}$.

Then $(Q; \mathrm{acl}^0)$ is a modular pregeometry.

Theorem 2.8. *Up to* acl^0*-interalgebraicity, any coherent* $P \subseteq \mathbb{K}^{\mathrm{eq}0}$ *is contained in a union of finitely many orthogonal such* Q*.*

Proof ingredients. Veblen-Young co-ordinatisation, plus generalisation of Evans-Hrushovski. □

3 Elekes-Szabó consequences

Definition 3.1. Say a finite subset X of a variety W is **τ-cgp** if for any proper subvariety $W' \subsetneq W$ of complexity $\leq \tau$, we have $|X \cap W'| < |X|^{\frac{1}{\tau}}$.

Definition 3.2. If $V \subseteq \prod_i W_i$ are irreducible complex algebraic varieties, with $\dim(W_i) = m$ and $\dim(V) = dm$, say V **admits a powersaving** if for some τ and $\epsilon > 0$

$$\left| \prod_i X_i \cap V \right| \leq O(N^{d-\epsilon})$$

for τ-cgp $X_i \subseteq W_i$ with $|X_i| \leq N$.

Lemma 3.3. *V admits no powersaving iff exists coherent generic $\overline{a} \in V(\mathbb{K})$.*

Definition 3.4. $H \leq G^n$ is a **special subgroup** if G is a commutative complex algebraic group and $H = \ker(A)^o$ for some $A \in \mathrm{Mat}(F \cap \mathrm{End}(G))$ for some division subalgebra $F \leq \mathrm{End}^0(G) := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(G)$.

Example 3.5. $G = \mathbb{C}^n$; then $\mathrm{End}^0(G) = \mathrm{End}(G) = \mathrm{Mat}_n(\mathbb{C})$, and $H \leq G^l$ is special iff it is a subspace defined by F-linear equations for some division subalgebra $F \leq \mathrm{Mat}_n(\mathbb{C})$.

Theorem 3.6. *$V \subseteq \prod_i W_i$ admits no powersaving iff it is in co-ordinatewise algebraic correspondence with a product of special subgroups.*

Where

Definition 3.7. Co-ordinatewise algebraic correspondence blah blah.

4 Applications

4.1 Generalised sum-product phenomenon

Corollary 4.1. *Let $(G_1, +_1)$ and $(G_2, +_2)$ be one-dimensional non-isogenous connected complex algebraic groups, and for $i = 1, 2$ let $f_i : G_i(\mathbb{C}) \to \mathbb{C}$ be a rational map. Then there are $\epsilon, c > 0$ such that if $A \subset \mathbb{C}$ is a finite set lying in the range of each f_i, then setting $A_i = f_i^{-1}(A) \subseteq G_i(\mathbb{C})$ we have*

$$\max(|A_1 +_1 A_1|, |A_2 +_2 A_2|) \geq c|A|^{1+\epsilon}.$$

Proof. Else,

$$\{(g_1, h_1, g_1 +_1 h_1, g_2, h_2, g_2 +_2 h_2) : f_1(g_1) = f_2(g_2), f_1(h_1) = f_2(h_2)\}$$

admits no powersaving, so get group $(G; +)$ such that Γ_{+_i} is in co-ordinatewise correspondence with Γ_+, $i = 1, 2$.

But then (by Ziegler) G_i is isogenous to G. □

Similarly in higher dimension, with a cgp assumption.

4.2 Intersections of varieties with approximate subgroups

Theorem 4.2. *$\Gamma \leq G(\mathbb{K})$ a $\emptyset$-$\bigwedge$-definable subgroup of a 1-dimensional algebraic group G, with $\boldsymbol{\delta}(\Gamma) = \dim(G)$.*

Then any coherent tuple $\overline{\gamma} \in \Gamma^n$ is generic in a coset of an algebraic subgroup of G^n.

Similarly in higher dimension, with a cgp assumption.

Corollary 4.3. *Let G be a commutative complex algebraic group. Suppose V is a subvariety of G^n which is not a coset of a subgroup. Then there are $N, \epsilon, \eta > 0$ depending only on G and the complexity of V such that if $A \subseteq G$ is a finite subset such that $A - A$ is τ-cgp and $|A + A| \leq |A|^{1+\epsilon}$ and $|A| \geq N$, then $|A^n \cap V| < |A|^{\frac{\dim(V)}{\dim(G)} - \eta}$.*

4.3 Diophantine connection

Example 4.4. $G = E$ complex elliptic curve.

$E[\infty] := \bigcup_m E[m]$ torsion subgroup.

Suppose $V \subseteq E^n$ is an irreducible closed complex subvariety such that $V(\mathbb{C}) \cap E[\infty]$ is Zariski dense in V. Let $d := \dim(V)$.

By Manin-Mumford, V is a coset of an algebraic subgroup. Hence for any $\epsilon > 0$, for arbitrarily large $r \in \mathbb{N}$,

$$|V(\mathbb{C}) \cap E[r!]^n| \geq |E[r!]|^{d-\epsilon}.$$

Suppose conversely that we only know this consequence of Manin-Mumford on the asymptotics of the number of torsion points in V. Then V has a coherent generic non-standard torsion point, and so by above theorem V is a coset.

Similarly for Mordell-Lang.

5 Sharpness

Consider case $G = \mathcal{G}_a^n$, so $\mathrm{End}^0(G) = \mathrm{End}(G) = \mathrm{Mat}_n(\mathbb{C})$.

Want to show: if $F \leq \mathrm{Mat}_n(\mathbb{C})$ is a division ring, and $H \leq G^l$ is a subspace defined by F-linear equations, then exists $\overline{\zeta} \in H$ coherent generic.

Fact 5.1 (Amitsur-Kaplansky)**.** *Any division subring $F \subseteq \mathrm{Mat}_n(\mathbb{C})$ has finite dimension over its centre.*

Corollary 5.2. *Exists finitely generated subring $\mathcal{O} \subseteq F$ such that H is defined by linear equations with coefficients from $\mathcal{O}$ and $\mathcal{O}$ is* ***constrainedly filtered****: there are finite $\mathcal{O}_n \subseteq \mathcal{O}$ such that*

(CF0) $\mathcal{O}_n \subseteq \mathcal{O}_{n+1}$; $\bigcup_{n\in\mathbb{N}} \mathcal{O}_n = \mathcal{O}$

(CF1) $\exists k.\ \forall n.\ \mathcal{O}_n + \mathcal{O}_n \subseteq \mathcal{O}_{n+k}$;

(CF2) $\forall a \in \mathcal{O}.\ \exists k.\ \forall n.\ a\mathcal{O}_n \subseteq \mathcal{O}_{n+k}$;

(CF3) $\forall \epsilon > 0.\ \frac{|\mathcal{O}_{n+1}|}{|\mathcal{O}_n|} \leq O(|\mathcal{O}_n|^\epsilon)$.

(e.g. $\mathbb{Z} = \bigcup_n [-2^n, 2^n]$ is constrainedly filtered.)

Let "$X_k := \prod_{s\to\mathcal{U}} (\sum_{i=1}^{s} \mathcal{O}_{s-k}\gamma_i)$" with $\gamma_i \in G$ generic independent.

Then $X := \bigcap_k X_k$ is an $\mathcal{O}$-submodule and $\boldsymbol{\delta}(X) = \boldsymbol{\delta}(X_0)$, and $\overline{\zeta} \in H \cap X^l$ with $\boldsymbol{\delta}(\overline{\zeta}) = \boldsymbol{\delta}(H \cap X^l)$ is coherent.

General G similar; always $\mathrm{End}^0(G)$ *embeds in* $\mathrm{Mat}^{\dim(G)}(\mathbb{C})$*, by Lie theory.*

6 Relaxing general position

Remark 6.1. $V :=$ graph of $(a_1, b_1) * (a_2, b_2) = (a_1 + a_2 + b_1^2 b_2^2, b_1 + b_2)$, $X_i := \{-N^4, \ldots, N^4\} \times \{-N, \ldots, N\} \subseteq \mathbb{C}^2 =: W_i$.

Then $|X_i^3 \cap V| \geq \Omega(|X_i|^2)$, but not in coarse general position, and V is not in co-ordinatewise correspondence with the graph of a group operation.