

# Universal covers of commutative finite Morley rank groups version f23cf3bf

Martin Bays, Bradd Hart, and Anand Pillay

Tue 19 Sep 20:24:33 CEST 2017

## Abstract

We give an algebraic description of the structure of the analytic universal cover of a complex abelian variety which suffices to determine the structure up to isomorphism. More generally, we classify the models of theories of “universal covers” of rigid divisible commutative finite Morley rank groups.

## 1 Introduction

### 1.1 Characterising universal covers of abelian varieties

Let  $\mathbb{G} = \mathbb{G}_m^n$  be a complex algebraic torus, or let  $\mathbb{G}$  be a complex abelian variety. Considering  $\mathbb{G}(\mathbb{C})$  as a complex Lie group, with  $L\mathbb{G} = T_0(\mathbb{G}(\mathbb{C}))$  its (abelian) Lie algebra, the exponential map provides a surjective analytic homomorphism

$$\exp : L\mathbb{G} \rightarrow \mathbb{G}(\mathbb{C}).$$

Let  $\mathcal{O} := \{\eta \in \text{End}(L\mathbb{G}) \mid \eta(\ker \exp) \subseteq \ker \exp\} \cong \text{End}(\mathbb{G})$  be the ring of  $\mathbb{C}$ -linear endomorphisms of  $L\mathbb{G}$  which induce endomorphisms of  $\mathbb{G}(\mathbb{C})$ ; these are precisely the algebraic endomorphisms of  $\mathbb{G}$ . Consider  $L\mathbb{G}$  as an  $\mathcal{O}$ -module.

In this paper, we use model theoretic techniques and Kummer theory to give a purely algebraic characterisation of the algebraic consequences of this analytic picture.

At first sight,  $\exp$  relates  $L\mathbb{G}$  to  $\mathbb{G}(\mathbb{C})$  in a rather particular way. For example, if  $a \in \mathbb{G}(\mathbb{C})$  and  $\exp(\alpha) = a$ , then  $\exp(\alpha/n)$  converges topologically to  $0 \in \mathbb{G}(\mathbb{C})$  - something which certainly needn't hold for an arbitrary  $\mathcal{O}$ -module homomorphism. We will show however that if we forget the topology and the analytic structure, leaving only the field structure on  $\mathbb{C}$  and the  $\mathcal{O}$ -module structure on  $L\mathbb{G}$ , and so work up to field automorphisms of  $\mathbb{C}$  and up to  $\mathcal{O}$ -module automorphisms of  $L\mathbb{G}$ , then  $\exp$  is distinguished from other  $\mathcal{O}$ -module homomorphisms only by its interaction with the torsion subgroup  $\mathbb{G}[\infty]$  of  $\mathbb{G}$ . More precisely, it is described by its restriction  $\exp|_{\langle \ker(\exp) \rangle_{\mathbb{Q}}} : \langle \ker(\exp) \rangle_{\mathbb{Q}} \rightarrow \mathbb{G}[\infty]$  to the divisible subgroup generated by  $\ker(\exp)$ : once this restriction is chosen, there is a unique way, up to automorphisms, to extend it to  $L\mathbb{G}$ .

**Theorem 1.1.** *Suppose  $\mathbb{G}$  and the action of each  $\eta \in \mathcal{O}$  are defined over a number field  $k_0 \leq \mathbb{C}$ .*

*Suppose  $\rho, \rho' : L\mathbb{G} \rightarrow \mathbb{G}(\mathbb{C})$  are surjective  $\mathcal{O}$ -module homomorphisms,  $\ker \rho' = \ker \rho$ , and  $\rho' \upharpoonright_{(\ker \rho')_{\mathcal{O}}} = \rho \upharpoonright_{(\ker \rho)_{\mathcal{O}}}$ .*

*Then there exists an  $\mathcal{O}$ -module automorphism  $\sigma \in \text{Aut}_{\mathcal{O}}(L\mathbb{G}/\ker \rho)$  and a field automorphism  $\tau \in \text{Aut}(\mathbb{C}/k_0)$  of  $\mathbb{C}$  fixing  $k_0$  such that the following diagram commutes, where  $\tau : \mathbb{G}(\mathbb{C}) \rightarrow \mathbb{G}(\mathbb{C})$  is the abstract group automorphism induced by  $\tau$ .*

$$\begin{array}{ccc} L\mathbb{G} & \xrightarrow{\sigma} & L\mathbb{G} \\ \downarrow \rho_1 & & \downarrow \rho_2 \\ \mathbb{G}(\mathbb{C}) & \xrightarrow{\tau} & \mathbb{G}(\mathbb{C}) \end{array}$$

We will define an  $\widehat{L}$ -isomorphism to be such a pair  $(\sigma, \tau)$  of an  $\mathcal{O}$ -module isomorphism and a field isomorphism which agree on  $\mathbb{G}$ . So Theorem 1.1 yields a characterisation of  $\exp : L\mathbb{G} \twoheadrightarrow \mathbb{G}(\mathbb{C})$ : it is, up to  $\widehat{L}$ -isomorphism, the unique surjective  $\mathcal{O}$ -homomorphism with its kernel and its restriction to the divisible subgroup generated by that kernel.

We require here that  $k_0$  is a number field in order to have Kummer theory available. We have a corresponding result in the case that  $\mathbb{G}$  is a split semiabelian variety defined over a number field, but general semiabelian varieties are problematic due to failure of Kummer theory.

We prove Theorem 1.1 by classifying the models of the first order theory of exp in an appropriate language  $\widehat{L}$ . Our proof can be split into three stages:

- (i) Kummer theory for abelian varieties explains the behaviour for finite extensions of  $k_0$ , and suffices to show uniqueness of the restriction of exp to  $\exp^{-1}(\mathbb{G}(\overline{\mathbb{Q}}))$ ;
- (ii) a function-field analogue of this Kummer theory allows us to extend the uniqueness to  $\mathbb{G}(F)$  for  $F$  an algebraically closed field of cardinality  $\leq \aleph_1$ ;
- (iii) we extend to arbitrary cardinals (in particular the continuum, which without assuming the continuum hypothesis is not covered by (ii)) using arguments involving independent systems, based on techniques involved in Shelah's Main Gap theorem.

In [BGH14], it was found that the geometric Kummer theory of (ii) actually follows from a general model-theoretic principle, Zilber's Indecomposability Theorem, and hence holds in the generality of rigid (see below) commutative divisible finite Morley rank groups.

This also turns out to be a natural level of generality for (iii), and it is in this context that we will actually work for most of this paper. We obtain an analogue of Theorem 1.1 in this generality, Theorem 3.31 below - although since there is no analogue of (i) in such generality we get a correspondingly weaker result.

This does allow us to remove the restrictions in Theorem 1.1 and still get a uniqueness result: if  $\mathbb{G}$  is an abelian variety over a field  $k_0 \leq \mathbb{C}$ , then the exponential map  $\exp : L\mathbb{G} \twoheadrightarrow \mathbb{G}(\mathbb{C})$  is, up to  $\widehat{L}$ -isomorphism fixing  $\exp^{-1}(\mathbb{G}(k_0^{\text{alg}}))$ , the unique surjective  $\text{End}(\mathbb{G})$ -homomorphism with kernel  $\ker \exp$  which extends

$\exp|_{\exp^{-1}(\mathbb{G}(k_0^{\text{alg}}))}$ . We obtain an analogous result for semiabelian varieties as part of Subsection 5.3.

We also obtain similar results for complex tori which are not abelian varieties, and for semiabelian varieties in positive characteristic, generalising [BZ11].

## 1.2 Profinite covers and an outline of the paper

For  $\mathbb{G} = \mathbb{G}(\mathbb{C})$  as above, or more generally for  $\mathbb{G}$  a commutative divisible finite Morley rank group, we associate a canonical structure  $\widehat{\mathbb{G}}$  which we call the “profinite universal cover” of  $\mathbb{G}$ , defined as the inverse limit of copies of  $\mathbb{G}$  with respect to the inverse system of multiplication-by- $n$  maps,  $\widehat{\mathbb{G}} := \varprojlim [n] : \mathbb{G} \rightarrow \mathbb{G}$ .

In the case of  $\mathbb{G}$  a complex semiabelian variety, this is the same construction that appears in the definition of the étale fundamental group - every finite étale cover of  $\mathbb{G}$  is dominated by some  $[n]$ , so taking the inverse limit with respect to all  $[n]$  amounts to taking the inverse limit with respect to all finite étale covers. So  $\widehat{\mathbb{G}}$  can be identified as the “étale universal cover” of  $\mathbb{G}$ .

In general, we can see  $\widehat{\mathbb{G}}$  as a purely algebraic substitute for an analytic universal cover of  $\mathbb{G}$ . We will see below in Remark 2.19 one justification for this: in an appropriate language  $\widehat{L}$ , if  $\mathbb{G}$  is a Lie group, then the Lie exponential map is an elementary submodel of the profinite universal cover  $\widehat{\mathbb{G}}$ .

The results described in the previous subsection result from classifying the models of the first-order theory of  $\widehat{\mathbb{G}}$ .

In Section 2 we define the structure we wish to consider on  $\widehat{\mathbb{G}}$ , axiomatise its first-order theory  $\widehat{T}$ , prove quantifier elimination, and examine it in terms of stability theory. In Section 3, we give a classification of the models of  $\widehat{T}$ . In Section 4, we return essentially to the context of Subsection 1.1, specialising the abstract model theory of earlier sections to the case of algebraic groups. Here we also use Kummer theory to strengthen the classification (peeking inside the prime model); the necessary Kummer theory is presented in Appendix A. Finally, in Section 5, we present some further natural examples of models of  $\widehat{T}$  for various  $\mathbb{G}$ , to which our classification theorem applies.

## 1.3 The literature

We discuss the previous work on which this work builds. For  $\mathbb{G} = \mathbb{G}_m$  the multiplicative group, Theorem 1.1 was proven in [Zil06] and [BZ11]. It was proven for  $\mathbb{G}$  an abelian variety in [Gav06] under the assumption of the continuum hypothesis, i.e. with only the first two of the three steps described above. A path to the full result was set out in [Zil02], and for  $\mathbb{G}$  an elliptic curve the full result was obtained in [Bay10].

These previous proofs of (iii) use algebraic techniques analogous to, but substantially more complicated and limited than, the model theoretic techniques of the present work. In previous work, the problem was considered one of categoricity in infinitary logic, and correspondingly the techniques applied were those of Shelah’s theory of excellent classes, and more specifically Zilber’s adaptation to Quasiminimal Excellent (QME) classes. It was key to the developments in this paper to instead consider the problem in terms of first-order classification theory. Although our results do not fall literally into the context of Shelah’s classification theory for superstable theories - essentially because we are inter-

ested in models where the kernel of  $\exp$  is rather unsaturated - and though ideas from the theory of Abstract Elementary Classes will still play a (largely implicit) role, the argument which allows us to get (iii) in the generality we do is an adaptation of Shelah’s “NOTOP” argument, which reduces the condition of excellence in the first-order case to a simpler condition.

In fact, while the current paper was in preparation, it was found that this same idea applies in the context of QME classes [BHH<sup>+</sup>14]. For the benefit of any readers familiar with that paper, we mention how it relates to this paper. Our main results do not fit into the definition of QME, even if we assume the kernel to be countable: we consider finite Morley rank groups which are not necessarily almost strongly minimal; correspondingly, the covers are not even almost quasiminimal. In the case discussed above of a semiabelian variety  $\mathbb{G}$ , however, the covers structure can be seen as almost quasiminimal - and moreover it is bi-interpretable with the quasiminimal structure induced on the inverse image in the cover of a Kummer-generic (in the sense of [BGH14]) curve in  $\mathbb{G}$  which generates  $\mathbb{G}$  as a group. So in this case, (iii) above could be deduced from the main result of [BHH<sup>+</sup>14].

## 1.4 Notation

We use unmarked tuple notation throughout: if  $A$  is a subset of a sort in a structure, we write  $x \in A$  if  $x$  is a finite tuple each co-ordinate of which is an element of  $A$ .

We write  $a \equiv_C b$  to mean that  $\text{tp}(a/C) = \text{tp}(b/C)$ , and we sometimes write  $\sigma : A \xrightarrow{\cong}_C B$  to denote that  $\sigma$  is an isomorphism which is the identity on  $C \subseteq A \cap B$ .

If  $G$  is an abelian group, we write  $G[n]$  for the  $n$ -torsion, and we write  $G[\infty]$  or  $\text{Tor}(G)$  for the torsion subgroup  $\bigcup_n G[n]$ .

We introduce further specialised notation in Section 2, after making relevant definitions.

## 1.5 Acknowledgements

This paper grew out of the thesis of the first author, supervised by Boris Zilber, and many of the ideas are due eventually to Zilber. The first author was also strongly influenced by the work of, and discussions with, Misha Gavrilovich. We would like to thank Udi Hrushovski for first suggesting the relevance of NOTOP/PMOP. The authors would also like to thank John Baldwin, Daniel Bertrand, Juan Diego Caycedo, Martin Hils and Jonathan Kirby for helpful pointers and discussion at various stages of this project.

We are grateful to the anonymous referee for an earlier version of the paper, whose suggestions and exhortations improved the paper substantially.

## 2 Profinite universal covers

In this section, we consider the algebra and basic model theory of our “profinite universal covers” of divisible commutative finite Morley rank groups.

## 2.1 $\widehat{G}$

We begin with some elementary definitions and remarks concerning abstract commutative groups.

If  $G$  is a commutative group and  $[n]$  is the multiplication-by- $n$  map, let  $\widehat{G}$  be the inverse limit  $\varprojlim [n] : G \rightarrow G$ . Let  $\rho_n : \widehat{G} \rightarrow G$  be the corresponding projections, so  $[n]\rho_{nm} = \rho_m$ . Let  $\rho := \rho_1$ . We often write elements of  $\widehat{G}$  in the form  $\gamma = (g_n)_n$ , so then  $\rho_n(\gamma) = g_n$ .

If  $\theta : G \rightarrow H$ , define  $\widehat{\theta} : \widehat{G} \rightarrow \widehat{H}$  by  $\widehat{\theta}((g_n)_n) = (\theta(g_n))_n$ .

**Definition 2.1.** The *divisible part*  $G^\circ$  of an abelian group  $G$  is the maximal divisible subgroup,  $G^\circ = \bigcap_{n>0} nG$ .

We will mostly work in contexts in which  $G^\circ$  is the ‘‘connected component’’ of  $G$  in one sense or another, hence the notation.

Say a commutative group  $G$  is *divisible-by-finite* if its divisible part  $G^\circ$  has finite index in  $G$ .

We note that  $\widehat{\phantom{x}}$  is an exact functor on divisible-by-finite groups:

**Lemma 2.2.** *Suppose  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence of divisible-by-finite groups. Then  $0 \rightarrow \widehat{A} \rightarrow \widehat{B} \rightarrow \widehat{C} \rightarrow 0$  is exact.*

*Proof.* Denote the given map  $B \rightarrow C$  as  $\theta$ . The only difficulty is the surjectivity of  $\widehat{\theta} : \widehat{B} \rightarrow \widehat{C}$ . We may assume  $A \rightarrow B$  is an inclusion. Factoring  $\theta$  via  $B/(A^\circ)$ , we see that it suffices to prove the surjectivity of  $\widehat{B} \rightarrow \widehat{C}$  under the assumption that  $A$  is divisible or finite.

- (a) Suppose  $A$  is divisible. We first show that given any  $n > 0$ ,  $b \in B$  and  $c' \in C$  such that  $\theta(b) = [n]c'$ , there is  $b' \in B$  such that  $[n]b' = b$  and  $\theta(b') = c'$ . Say  $\theta(b'') = c'$ ; then  $\theta([n]b'') = [n]c' = \theta(b)$ , so  $b - [n]b'' \in A$ . Say  $a' \in A$  with  $[n]a' = b - [n]b''$ . Then  $b' := b'' + a'$  is as required.

Given  $\widehat{c}$ , we can therefore inductively define  $b_{n!}$  such that  $[n+1]b_{(n+1)!} = b_{n!}$  and  $\theta(b_{n!}) = \rho_{n!}(\widehat{c})$ . Easily, there is a unique  $\widehat{b} \in \widehat{B}$  such that  $\rho_{n!}(\widehat{b}) = b_{n!}$ , and it satisfies  $\widehat{\theta}(\widehat{b}) = \widehat{c}$ .

- (b) Suppose  $A$  is finite, say  $[n]A = 0$ . Then  $\theta$  factors  $[n]$  - indeed, let  $\phi$  be the map making the left triangle in the following diagram commutes, then note that the right triangle also commutes. But  $\widehat{[n]}$  is surjective, hence so is  $\widehat{\theta}$ .

$$\begin{array}{ccc}
 B & \xrightarrow{[n]} & B \\
 \theta \searrow & \phi \nearrow & \searrow \theta \\
 & C & \xrightarrow{[n]} C
 \end{array}$$

□

## 2.2 $\widehat{T}$

Now let  $\mathbb{G}$  be a connected commutative finite Morley rank group, and suppose moreover that it is divisible. Then  $[n] : \mathbb{G} \rightarrow \mathbb{G}$  has finite kernel, and it follows

that any definable subgroup  $A \leq \mathbb{G}$  is divisible-by-finite, and its divisible part  $A^o$  is its connected component in the model-theoretic sense, namely the smallest definable subgroup of finite index.

Let  $T := \text{Th}(\mathbb{G})$ ; we assume (by appropriate choice of language) that  $T$  has quantifier elimination. We also assume that the language  $L$  of  $T$  is countable.

Let  $\widehat{T}$  be the theory of  $(\widehat{\mathbb{G}}, \mathbb{G})$  in the two-sorted language  $\widehat{L}$  consisting of the maps  $\rho_n$  for each  $n$ , the full  $T$ -structure on  $\mathbb{G}$ , and, for each  $\text{acl}^{\text{eq}}(\emptyset)$ -definable connected subgroup  $H$  of  $\mathbb{G}^n$ , a predicate  $\widehat{H}$  interpreted as the subgroup  $\widehat{H} = \{x \mid \bigwedge_n \rho_n(x) \in H\}$  of  $\widehat{\mathbb{G}}^n$ . We will see below that  $\widehat{T}$  depends only on  $T$ .

For quantifier elimination purposes, we actually assume (by expanding  $T$  by constants if necessary) that every  $\text{acl}^{\text{eq}}(\emptyset)$ -definable connected subgroup of  $\mathbb{G}^n$  is  $\emptyset$ -definable.

We say that  $T$  is *rigid* if for  $\mathbb{G}$  a saturated model of  $T$ , every definable connected subgroup of  $\mathbb{G}^n$  is defined over  $\text{acl}^{\text{eq}}(\emptyset)$ . Although the results of this section do not require rigidity, our language is chosen with it in mind.

*Remark 2.3.* As in the proof of Lemma 2.2, any definable finite group cover of  $\mathbb{G}$  is dominated by some  $[n]$ , so is “seen” by  $\widehat{\mathbb{G}}$ .

Note that divisibility is crucial for this - for example, the Artin-Schreier map  $x \mapsto x^p - x$  is a finite definable group cover of the additive group in  $\text{ACF}_p$  which isn't handled by our setup (c.f. [BGH14] where this issue is discussed).

**Notation 2.4.** Suppose  $(\widetilde{M}, M)$  is an  $\widehat{L}$ -structure.

- If  $\widetilde{a} \in \widetilde{M}$  is a tuple, then we will write  $\widehat{a}_n$  for  $\rho_n(\widetilde{a})$ , and  $\widehat{a}$  for  $\rho(\widetilde{a})$ , and  $\widehat{a}$  for  $(\widehat{a}_n)_n$ . Similarly, if  $\widetilde{A} \subseteq \widetilde{M}$ , we write  $\widehat{A}$  for  $\bigcup_n \rho_n(\widetilde{A})$ .
- We will usually just write  $\widetilde{M} \models \widehat{T}$  to mean  $(\widetilde{M}, M) \models \widehat{T}$ .
- $\widehat{G}$  and  $\widehat{H}$  will always denote the predicates corresponding to  $\emptyset$ -definable connected subgroups  $G$  and  $H$  of a cartesian power of  $\mathbb{G}$ .  $\widehat{C}$  will denote a coset of some  $\widehat{H}$ .
- $\widehat{G}(\widetilde{a})$  is the definable set  $\{\widetilde{x} \mid (\widetilde{x}, \widetilde{a}) \in \widehat{G}\}$ , a coset of  $\widehat{G}(0)$ . Similarly for  $G(\widehat{a})$ .
- $\ker$  is the definable set  $\ker(\rho)$ .
- $\ker^o$ , the divisible part of  $\ker$ , is the  $\bigwedge$ -definable set  $\bigwedge_n \rho_n(x) = 0$ .
- Abusively,  $\ker$  and  $\ker^o$  also refer to the corresponding sets in cartesian powers of  $\mathbb{G}$ .
- If  $\text{pr} : \widetilde{M}^n \rightarrow \widetilde{M}^m$  is a co-ordinate projection, we also write  $\text{pr}$  for the corresponding co-ordinate projection  $M^n \rightarrow M^m$ , and we also write  $\text{pr}$  for the restriction of  $\text{pr}$  to a subset  $\widetilde{A} \subseteq \widetilde{M}^n$  or  $A \subseteq M^n$ , leaving it to context to disambiguate.
- $\widehat{H}_0 := \widehat{H} \cap \ker^o$ , a  $\mathbb{Q}$ -subspace of the  $\mathbb{Q}$ -vector space  $\ker^o$ .

### 2.3 Axiomatisation and quantifier elimination

We now give a list of first order axioms for a structure  $\widetilde{M}$  in the language of  $\widehat{T}$ . We show in Proposition 2.8 that these axioms axiomatise  $\widehat{T}$ .

#### Axioms 2.5.

- (A1)  $M \models T$
- (A2) Let  $\Gamma_+$  be the graph of the group operation on  $\mathbb{G}$ . Then  $\widehat{\Gamma}_+$  is the graph of a commutative divisible torsion free group operation, which we write as “+” and work with respect to in the following axioms;
- (A3) Let  $\Delta$  be the diagonal subgroup of  $\mathbb{G}$ , i.e. the graph of equality. Then  $\widehat{\Delta}$  is the diagonal subgroup of  $\widetilde{M}$ .
- (A4) Each  $\widehat{H}$  is a divisible subgroup.
- (A5)  $[m]\rho_{nm} = \rho_n$ .
- (A6)  $\rho_n(\widehat{H}) = H$ .
- (A7)  $\widehat{G} \cap \widehat{H} = \widehat{H}'^o$  where  $H' := G \cap H$ .
- (A8) If  $H \subseteq G$  and  $\text{Tor}(H) = \text{Tor}(G)$ , then  $\widehat{H} \cap \ker = \widehat{G} \cap \ker$ .
- (A9) If a co-ordinate projection  $\text{pr}$  induces a surjection  $\text{pr} : G \rightarrow H$  with kernel  $K$  then the corresponding co-ordinate projection induces a surjection  $\text{pr} : \widehat{G} \rightarrow \widehat{H}$  with kernel  $\widehat{K}^o$ .

In  $\widehat{\mathbb{G}}$  and other models of  $\widehat{T}$  which we will be considering in the applications,  $\widehat{H}$  will be the divisible part of  $\rho^{-1}(H)$ . In this case, the following lemma substantially simplifies verification of the axioms.

**Lemma 2.6.** *Suppose  $V$  is a divisible torsion-free abelian group, and  $\rho : V \rightarrow \mathbb{G}$  is a surjective homomorphism. For  $H$  a connected definable subgroup of  $\mathbb{G}^n$ , let  $\widehat{H}$  be the divisible part of  $\rho^{-1}(H)$ . Suppose that  $\ker$  has trivial divisible part (i.e.  $\widehat{0} = 0$ ). Let  $\rho_n(x) := \rho(x/n)$ . Then with this structure,  $V$  satisfies (A1)-(A9) if it satisfies (A6) and (A9).*

*Proof.* (A1) Immediate.

(A2)  $\widehat{\Gamma}_+ = \{(x, y, z) \mid \forall n. x/n + y/n - z/n \in \ker\}$ , which, since  $\ker$  has trivial divisible part, is the graph of + on  $V$ .

(A3) Similar.

(A4) Immediate from the definition of  $\widehat{H}$ .

(A5) Immediate from the definition of  $\rho_n$ .

(A6) Assumed.

(A7)  $\widehat{G} \cap \widehat{H}$  is a divisible subgroup of  $\rho^{-1}(H'^o)$  (where  $H' = G \cap H$ ), so is contained in  $\widehat{H}'^o$ . Similarly for the converse inclusion.

(A8) Suppose  $H \subseteq G$ ,  $\zeta \in \widehat{G} \cap \ker$ , and  $\text{Tor}(G) = \text{Tor}(H)$ . Then  $\mathbb{Q}\zeta \subseteq \rho^{-1}(H)$ , so  $\zeta \in \widehat{H}$  by definition of  $\widehat{H}$ .

(A9) Assumed. □

**Lemma 2.7.**  $\widehat{\mathbb{G}}$  satisfies the axioms (A1)-(A9).

*Proof.* We appeal to Lemma 2.6. (A6) and the fact that  $\widehat{H}$  is the connected component of  $\rho^{-1}(H)$  are immediate from the definitions. (A9) follows from Lemma 2.2. □

**Proposition 2.8.** (A1)-(A9) axiomatise  $\widehat{T}$ , and  $\widehat{T}$  has quantifier elimination.

*Proof.* Let  $\widehat{T}'$  be the theory axiomatised by (A1)-(A9).

We show that  $\widehat{T}'$  is complete and admits quantifier elimination. Completeness and Lemma 2.7 then implies that  $\widehat{T}' = \widehat{T}$ .

We first note some elementary deductions from the axioms:

(D1) For any  $H$  and  $G$ , we have by (A9) applied to  $\text{pr} : H \times G \twoheadrightarrow G$  that  $\widehat{H \times G} = \widehat{H} \times \widehat{G}$ .

(D2) By (A6) applied to the graph of the group operation, the  $\rho_n$  are homomorphisms.

(D3) In the context of (A9), if  $K/K^\circ$  has exponent  $e$ , then  $e \cdot (\widehat{H} \cap \ker) \subseteq \text{pr}(\widehat{G} \cap \ker)$ . Indeed, this follows from (A9), (A6), and the snake lemma applied to the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{K^\circ} & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{H} & \longrightarrow & 0 \\ & & \downarrow \rho & & \downarrow \rho & & \downarrow \rho & & \\ 0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

Now suppose we have  $\omega$ -saturated models  $\widetilde{M}, \widetilde{N} \models \widehat{T}'$ , finite tuples  $\widetilde{m} \equiv_{qf} \widetilde{n}$  from each with equal quantifier-free types, and a point  $\widetilde{m}' \in \widetilde{M}$ . To conclude the proof, we must find  $\widetilde{n}' \in \widetilde{N}$  such that  $(\widetilde{m}, \widetilde{m}') \equiv_{qf} (\widetilde{n}, \widetilde{n}')$ .

Let  $\widehat{H}$  be least such that it contains  $\widetilde{m}$ . This exists by  $\omega$ -stability of  $T$  and (A7); c.f. Definition 2.13 below.

Let  $\widehat{G}$  be least such that it contains  $(\widetilde{m}, \widetilde{m}')$ , and let  $\text{pr} : (\widetilde{m}, \widetilde{m}') \mapsto \widetilde{m}$  be the co-ordinate projection.

We work in  $\widehat{T}'$ ; when we make a statement which is expressible as a sentence in  $\widehat{L}$ , we mean that it is a consequence of  $\widehat{T}'$ .

$\text{pr}(\widehat{G}) = \widehat{\text{pr}(G)}$  by (A9), so  $\widehat{H} \subseteq \text{pr}(\widehat{G})$ , and  $\text{pr}^{-1}(\widehat{H}) = \widehat{H} \times \widehat{\mathbb{G}} = \widehat{H \times \mathbb{G}} = \widehat{\text{pr}^{-1}(H)}$ , so  $\widehat{G} \subseteq \text{pr}^{-1}(\widehat{H})$  and so  $\text{pr}(\widehat{G}) \subseteq \widehat{H}$ . So  $\widehat{H} = \text{pr}(\widehat{G})$ , and so  $\text{pr} : \widehat{G} \twoheadrightarrow \widehat{H}$  and  $\text{pr} : G \twoheadrightarrow H$ .

**Claim 2.9.**  $\text{pr} : \widehat{G}_0(\widetilde{N}) \twoheadrightarrow \widehat{H}_0(\widetilde{N})$



*Proof.* Work in  $\tilde{N}$ . Let  $K$  be the kernel of  $\text{pr} : G \rightarrow H$ , and suppose  $K/K^o$  has exponent  $e$ , so by (D3),  $\text{pr} : \widehat{G} \cap \ker \rightarrow e(\widehat{H} \cap \ker)$ . So for each  $k$ ,

$$\text{pr} : k(\widehat{G} \cap \ker) \rightarrow ke(\widehat{H} \cap \ker).$$

But then by  $\omega$ -saturation of  $\tilde{N}$ ,

$$\text{pr} : \widehat{G}_0 = \bigcap_k k(\widehat{G} \cap \ker) \rightarrow \bigcap_k ke(\widehat{H} \cap \ker) = \widehat{H}_0.$$

□

By QE in  $T$  and  $\omega$ -saturation, we can find  $\tilde{n}' \in \tilde{N}$  such that

$$(\tilde{m}, \tilde{m}') \equiv_{qf} (\tilde{n}, \tilde{n}') \quad (*)$$

as infinite tuples; in particular,  $\rho_k(\tilde{n}, \tilde{n}') \in G$  for all  $k$ , and so by  $\omega$ -saturation we find  $\tilde{n}'' \in \widehat{G}(\tilde{N})$  such that  $\rho_k(\tilde{n}, \tilde{n}') = \rho_k(\tilde{n}'')$  for all  $k$ , and so  $\tilde{\zeta} := \tilde{n}'' - (\tilde{n}, \tilde{n}') \in \ker^0(\tilde{N})$ . Then  $\text{pr} \tilde{\zeta} \in \widehat{H}_0(\tilde{N})$ , and so by the Claim there is  $\tilde{\zeta}' \in \widehat{G}_0(\tilde{N})$  with  $\text{pr} \tilde{\zeta}' = \text{pr} \tilde{\zeta}$ , and then  $\tilde{n}'' + \tilde{\zeta}' \in \widehat{G}$  and  $\text{pr}(\tilde{n}'' + \tilde{\zeta}') = \tilde{n}$ .

So we can assume  $(\tilde{n}, \tilde{n}') \in \widehat{G}$ , while still satisfying (\*). Now suppose  $(\tilde{n}, \tilde{n}')$  is contained in a proper subgroup  $\widehat{G}' < \widehat{G}$ . Then  $\rho_k(\tilde{m}, \tilde{m}') \in G'$  for each  $k$ , so by  $\omega$ -saturation,  $(\tilde{m}, \tilde{m}') \in \widehat{G}' + \tilde{\zeta}$  for some  $\tilde{\zeta} \in \widehat{G}_0 \setminus \widehat{G}'_0$ . So  $\widehat{G}'_0(\tilde{M}) < \widehat{G}_0(\tilde{M})$ , so, by (A8),  $\text{Tor}(G') < \text{Tor}(G)$ . Hence by (A6) and (A5), for each  $k$  there is  $\tilde{\zeta} \in \widehat{G} \setminus \widehat{G}'$  with  $\rho_k(\tilde{\zeta}) = 0$ , and so by saturation  $\widehat{G}'_0(\tilde{N}) < \widehat{G}_0(\tilde{N})$ .

Now  $\text{pr}(\widehat{G}') = \widehat{H}$ , by the same argument which showed  $\text{pr}(\widehat{G}) = \widehat{H}$ , and so the Claim applies also to  $\widehat{G}'$ . So  $\text{pr}(\widehat{G}'_0(\tilde{N})) = \widehat{H}_0(\tilde{N}) = \text{pr}(\widehat{G}_0(\tilde{N}))$ . Hence we have a strict inclusion  $\widehat{G}'_0(0) < \widehat{G}_0(0)$  in  $\tilde{N}$  for the fibres above  $0 \in \widehat{H}$ . So by translating, we can find  $\tilde{n}'$  satisfying (\*) and such that  $(\tilde{n}, \tilde{n}') \notin \widehat{G}'$ . Now  $\widehat{G}_0(0)$  is not covered by any finitely many such  $\widehat{G}'_0(0)$ , since they are proper  $\mathbb{Q}$ -subspaces. So we can avoid any finitely many such proper subgroups simultaneously, and so by  $\omega$ -saturation, we find  $\tilde{n}'$  satisfying (\*) for which  $\widehat{G}$  is least such that it contains  $(\tilde{n}, \tilde{n}')$ .

It follows, using (A3) for formulae involving equality on the sort  $\widehat{\mathbb{G}}$ , that  $(\tilde{m}, \tilde{m}') \equiv_{qf} (\tilde{n}, \tilde{n}')$  as required. □

*Remark 2.10.* Assuming that  $T$  has finite Morley rank is a much stronger assumption than we need for this result. Really the result is about the reduct to the abelian structure of  $\mathbb{G}$  with predicates for the  $\text{acl}^{\text{eq}}(\emptyset)$ -definable subgroups of  $\mathbb{G}^n$ ; all we require is that these subgroups have divisible definable connected components, and the descending chain condition on definable subgroups. For example,  $\mathbb{G}$  could be real semiabelian variety  $S(\mathbb{R})$  with the semialgebraic structure of its interpretation in the real field.

*Remark 2.11.* The assumption that each  $\text{acl}^{\text{eq}}(\emptyset)$ -definable connected subgroup  $H$  is actually  $\emptyset$ -definable in  $T$  is necessary, because  $H$  is  $\emptyset$ -definable in  $\widehat{T}$  as the image of  $\widehat{H}$ , while quantifier elimination implies that  $\mathbb{G}$  has only the structure of  $T$ .

**Corollary 2.12.** *Suppose  $\tilde{B} \subseteq \tilde{M} \models \hat{T}$ , and suppose  $X \subseteq \tilde{M}^n$  is definable over  $\tilde{B}$ . There are  $H_i, \tilde{b}^i \in \tilde{B}$ ,  $m > 0$ , and  $\emptyset \neq Y_i \subseteq H_i(\tilde{b}_m^i)$ , with  $i$  ranging through a finite set, and with each  $Y_i$  being  $T$ -definable over  $\tilde{B}$ , such that*

$$\bigcup_i \left( \hat{H}_i(\tilde{b}^i) \cap \rho_m^{-1}(Y_i) \right) \subseteq X \subseteq \bigcup_i \hat{H}_i(\tilde{b}^i).$$

*Proof.* This follows from the QE, using (A7) to reduce an intersection of cosets to a single coset, using (A5) to reduce to a single  $\rho_m$ , and using that (by (A6))  $\rho_m(\hat{H}(\tilde{b})) \subseteq H(b_m)$ .  $\square$

**Definition 2.13.** Let  $\tilde{B} \subseteq \tilde{M} \models \hat{T}$ , and  $\tilde{a} \in \tilde{M}$ . Then  $\text{grploc}(\tilde{a}/\tilde{B})$ , the *group locus* of  $\tilde{a}$  over  $\tilde{B}$ , is the smallest set containing  $\tilde{a}$  of the form  $\hat{H}(\tilde{b})$  with  $\tilde{b} \in \tilde{B}$ .

*Remark 2.14.* Such a smallest set exists, by (A7) and  $\omega$ -stability of  $T$ .

Clearly  $\text{grploc}(\tilde{a}/\tilde{B})$  is definable over  $\tilde{B}$ ; however, it is not true that  $\text{grploc}(\tilde{a}/\tilde{B})$  is necessarily the smallest coset of a  $\hat{G}$  containing  $\tilde{a}$  which is definable over  $\tilde{B}$ . For example, suppose  $\mathbb{G}$  is a torsion-free group, so  $\rho$  is an isomorphism, and consider a coset  $\hat{G} + \tilde{a}$  with  $a \in \text{dcl}^T(B) \setminus B$ .

*Remark 2.15.* Using (A2), (A7), and (A9), we see that  $\hat{H}(\tilde{b} + \tilde{b}')$  can be rewritten in the form  $\hat{G}(\tilde{b}, \tilde{b}')$ , and similarly for  $\hat{H}(\tilde{b}) + \tilde{b}'$ . So in particular,  $\text{grploc}(\tilde{a}/\tilde{B}) = \text{grploc}(\tilde{a}/\langle \tilde{B} \rangle)$  where  $\langle \tilde{B} \rangle$  is the subgroup of  $\tilde{M}$  generated by  $\tilde{B}$ .

**Lemma 2.16.** *Let  $\tilde{B} \subseteq \tilde{M} \models \hat{T}$ , and  $\tilde{a} \in \tilde{M}$ . Let  $\hat{C} := \text{grploc}(\tilde{a}/\tilde{B})$ .*

*Suppose  $\ker(\tilde{M}) \subseteq \tilde{B}$ .*

*Then  $p'(\tilde{x}) := \text{tp}(\tilde{a}/\hat{B}) \cup \{\tilde{x} \in \hat{C}\} \models \text{tp}(\tilde{a}/\tilde{B})$*

*Proof.* By the QE, we need only see that if  $\tilde{a}' \models p'$  in an elementary extension, then for all  $\hat{H}$  and all  $\tilde{b} \in \tilde{B}$ ,  $\tilde{a} \in \hat{H}(\tilde{b})$  iff  $\tilde{a}' \in \hat{H}(\tilde{b})$ .

Now  $\tilde{a} \in \hat{H}(\tilde{b})$  iff  $\hat{C} \leq \hat{H}(\tilde{b})$ , so the forward direction is clear.

For the converse, suppose  $\tilde{a}' \in \hat{H}(\tilde{b})$ . Then  $\tilde{a}' \in H(b)$ , hence  $\tilde{a} \in H(b)$ . So  $(\tilde{a}, \tilde{b}) \in \hat{H} + \ker(\tilde{M})$ , i.e.  $\tilde{a} \in \hat{H}(\tilde{b} + \tilde{\zeta}) + \tilde{\xi}$  for some  $\tilde{\zeta}, \tilde{\xi} \in \ker(\tilde{M})$ . But  $\ker(\tilde{M}) \subseteq \tilde{B}$ , so by Remark 2.15,  $\hat{C} \leq \hat{H}(\tilde{b} + \tilde{\zeta}) + \tilde{\xi}$ . So  $\tilde{a}' \in \hat{H}(\tilde{b}) \cap (\hat{H}(\tilde{b} + \tilde{\zeta}) + \tilde{\xi})$ ; but this is an intersection of cosets of  $\hat{H}(0)$ , so they are equal, and so  $\tilde{a} \in \hat{H}(\tilde{b})$ .  $\square$

*Remark 2.17.* It also follows from the QE that  $\ker^0$  is indeed the connected component of the kernel in the model-theoretic sense, and more generally that  $\hat{H} + \ker^0$  is the connected component of  $\rho^{-1}(H) = \hat{H} + \ker$ .

## 2.4 Lie exponential maps as models of $\hat{T}$

Let  $\mathbb{G}$  be a connected commutative Lie group which is also equipped with a finite Morley rank group structure for which the model-theoretically connected definable subgroups of  $\mathbb{G}^n$  are topologically connected closed Lie subgroups. This is the case for a connected commutative complex algebraic group  $\mathbb{G}(\mathbb{C})$  with the Zariski structure, and we will discuss other examples in Section 5.

Consider the Lie algebra  $L\mathbb{G} = T_0\mathbb{G}$  with the Lie exponential map  $\exp : L\mathbb{G} \rightarrow \mathbb{G}$  as an  $\hat{L}$ -structure, with  $\rho_m(x) := \exp(x/m)$  and  $\hat{H} := LH \leq L\mathbb{G}^n$  for  $H \leq \mathbb{G}^n$  connected definable.

**Proposition 2.18.**  $L\mathbb{G} \models \widehat{T}$ .

*Proof.* We appeal to Lemma 2.6.

(A6) holds since  $\exp$  is surjective for commutative Lie groups (since the image is a subgroup which contains a neighbourhood of the identity).

So since  $LH$  is divisible and  $\ker \exp$  is discrete,  $\widehat{H} = LH$  is the divisible part of  $\rho^{-1}(H)$ .

Finally, (A9) follows from exactness of the functor  $L$  for commutative Lie groups. To check this in the setting of (A9), the only difficulty is the surjectivity of  $L\mathbb{G} \rightarrow LH$ , but this follows from the fact that the image is an  $\mathbb{R}$ -vector subspace of dimension  $\dim(L\mathbb{G}) - \dim(LK) = \dim(G) - \dim(K) = \dim(H) = \dim(LH)$ .  $\square$

*Remark 2.19.* Note that  $x \mapsto (\exp(x/n))_n$  is an embedding of  $L\mathbb{G}$  into  $\widehat{\mathbb{G}}$ , which, by the QE, is elementary.

*Remark 2.20.* Lie theory provides a topological interpretation of the embedding of Remark 2.19.

The group  $\widehat{\mathbb{G}}$  is easily seen to be isomorphic to the group of abstract group homomorphisms  $\text{Hom}(\mathbb{Q}, \mathbb{G})$ , by taking the image in  $\widehat{\mathbb{G}}$  of  $\theta \in \text{Hom}(\mathbb{Q}, \mathbb{G})$  to be  $(\theta(1/n))_n$ . Then by recalling that  $x \mapsto (t \mapsto \exp(tx))$  is an isomorphism of  $L\mathbb{G}$  with the group  $\text{Hom}_c(\mathbb{R}, \mathbb{G})$  of 1-parameter subgroups, and considering their restrictions to  $\mathbb{Q}$ , we see that the image in  $\text{Hom}(\mathbb{Q}, \mathbb{G})$  of  $L\mathbb{G}$  is precisely the subgroup  $\text{Hom}_c(\mathbb{Q}, \mathbb{G})$  of continuous homomorphisms.

By translation,  $\theta \in \text{Hom}(\mathbb{Q}, \mathbb{G})$  is continuous iff it is continuous at 0, which holds iff  $\lim_{n \rightarrow \infty} \theta(1/n) = 0 \in \mathbb{G}$ , which holds iff this limit exists. So we can also identify  $L\mathbb{G}$  as the subgroup of convergent elements of  $\widehat{\mathbb{G}}$ , when viewed as sequences  $(a_n)_n$ .

## 2.5 Stability theory of $\widehat{T}$

**Proposition 2.21.** (i)  $\widehat{T}$  is superstable.

(ii) If  $\text{tp}(\tilde{a}/\tilde{B})$  forks over  $\tilde{A} \subseteq \tilde{B}$  then either  $\text{tp}(a/\widehat{B})$  forks over  $\widehat{A}$  or  $\text{grploc}(\tilde{a}/\tilde{B})$  is not definable over  $\tilde{A}$ .

(iii)  $\widehat{T}$  has finite  $U$ -rank, i.e.  $U(\tilde{a}/\tilde{B}) < \omega$  for any  $\tilde{a}, \tilde{B}$ .

*Proof.* (i) By the QE,  $\text{tp}(\tilde{a}/\tilde{A})$  is determined by  $\text{tp}(\widehat{a}/\widehat{A})$  and  $\text{grploc}(\tilde{a}/\tilde{A})$ .

The former is determined by  $\text{tp}(a/\widehat{A})$  and  $(\text{tp}(a_k/\widehat{A}a))_k$ , and since  $[k]$  has finite kernel there are only finitely many possibilities for each  $\text{tp}(a_k/\widehat{A}a)$ .  $\text{grploc}(\tilde{a}/\tilde{A})$  is determined by a choice of coset over  $\tilde{A}$ . So by  $\omega$ -stability of  $T$ , if  $|\tilde{A}| = \lambda \geq 2^{|\tilde{T}|}$  then  $|S(\tilde{A})| \leq (\lambda 2^{8\omega})(|\tilde{T}|\lambda) = \lambda$ . So  $\widehat{T}$  is superstable.

(ii) Suppose  $\text{tp}(\tilde{a}/\tilde{B})$  forks over  $\tilde{A}$ ; say  $\phi(x, \tilde{b}) \in \text{tp}(\tilde{a}/\tilde{B})$  divides over  $\tilde{A}$ . Let  $\widehat{C} := \text{grploc}(\tilde{a}/\tilde{B})$ . We may assume  $\phi(x, \tilde{b}) \models x \in \widehat{C}$ .

Suppose  $\widehat{C}$  is defined over  $\tilde{A}$ . Then also  $\phi(x, \tilde{b}') \models x \in \widehat{C}$  for any  $\tilde{b}' \equiv_{\tilde{A}} \tilde{b}$ . Now by Corollary 2.12,  $\phi(x, \tilde{b})$  is implied by a formula in  $\text{tp}(\tilde{a}/\tilde{B})$  of the form

$$x \in \widehat{C} \wedge \psi(\rho_n(x))$$

where  $\psi(x)$  is a  $T$ -formula over  $\widehat{B}$  implying  $x \in \rho_n(\widehat{C})$ . So since  $\phi$  divides over  $\widehat{A}$ ,  $\psi$  must divide over  $\widehat{A}$ . So  $\text{tp}(a_n/\widehat{B})$  forks over  $\widehat{A}$ , and since  $a$  is algebraic over  $a_n$ , so does  $\text{tp}(a/\widehat{B})$ .

- (iii) Finite rankedness of  $\widehat{T}$  follows from (ii), finite rankedness of  $T$ , and the fact that Morley rank bounds the length of chains of connected subgroups in  $T$ . □

We end this section with a proposition giving a stability-theoretic analysis of  $\widehat{T}$ ; these results are not used explicitly in the remainder of the paper, but they inform it.

**Proposition 2.22.** *Let  $\widetilde{\mathcal{C}} \models \widehat{T}$  be a monster model.*

- (i)  $\ker^0$  is stably embedded, in the sense that every relatively definable set is relatively definable with parameters from  $\ker^0$ . Consider  $\ker^0(\widetilde{\mathcal{C}})$  as a structure with the  $\emptyset$ -relatively-definable sets as predicates, and let  $\widehat{T}^0 := \text{Th}(\ker^0(\widetilde{\mathcal{C}}))$ . Then  $\widehat{T}^0$  is an  $\omega$ -stable 1-based group of finite Morley rank bounded above by the Morley rank of  $T$ .

In particular,  $\ker^0$  has finite relative Morley rank in the sense of [BBP16].

- (ii)  $\text{im}(\rho)$  is stably embedded with induced structure precisely that of  $T$ .

- (iii) Every type in  $\widehat{T}^{\text{eq}}$  is analysable in  $\ker^0$  and  $\text{im}(\rho)$ .

- (iv)  $\ker^0$  is orthogonal to  $\text{im}(\rho)$ .

- (v) A regular type in  $\widehat{T}^{\text{eq}}$  is non-orthogonal to one of

(a) a strongly minimal type in  $T^{\text{eq}}$ ;

(b)  $\widehat{G}_0/\widehat{H}_0$  where  $H \leq G$  have no intermediate connected subgroup.

- (vi)  $\widehat{T}$  has weak elimination of imaginaries in  $T^{\text{eq}}$  and the sorts  $\widetilde{\mathcal{C}}^n/\widehat{H}$ .

*Proof.* (i) By the QE, the only structure on  $\ker^0$  is the abelian structure given by the  $\widehat{H}_0$ . Stable embeddedness and 1-basedness follow easily. (Stable embeddedness can alternatively be deduced directly from stability of  $\widehat{T}^0$ .)

Since  $\ker^0$  is torsion-free and  $\widehat{H} \cap \widehat{G} = \widehat{H}'$  where  $H' = (H \cap G)^\circ$ , the definable subgroups are precisely those of the form  $\widehat{H}_0$ . So there is no infinite chain of definable subgroups of  $\ker^0$ , so  $\widehat{T}^0$  is of finite Morley rank. The rank is bounded by the longest length of such a chain, which is bounded by the rank of  $T$ .

- (ii) This is immediate from the QE.

- (iii) Consider a strong type  $q = \text{stp}(\widetilde{a}/A)$ , with  $A \subseteq \mathcal{C}^{\text{eq}}$ . If  $\widetilde{b} \in \widetilde{\mathcal{C}}$  is a realisation of  $q_1 = \text{stp}(\widetilde{a}/aA)$  independent from  $\widetilde{a}$  over  $A$ , then since  $\widehat{a} \subseteq \text{acl}(a)$ , we have  $\widetilde{a} - \widetilde{b} \in \ker^0$ . So  $q_1$  is internal to  $\ker^0$ , and clearly  $\text{stp}(a/A)$  is internal to  $\text{im}(\rho)$ .

- (iv) It is immediate from the QE that every relatively definable subset of  $(\ker^0)^n \times \text{im}(\rho)^m$  is a Boolean combination of products of subsets of  $(\ker^0)^n$  with subsets of  $\text{im}(\rho)^m$ .
- (v) By (i), the types in (b) are minimal, and  $\ker^0$  is analysed in them. So this follows from (iii).
- (vi) For  $\phi(x, y)$  an atomic formula, it is easy to see that any  $\phi$ -type over a model has canonical parameter in these sorts. So by the QE, any type over a model has canonical base in these sorts. By stability, the same holds for any type over an  $\text{acl}^{\text{eq}}$ -closed set. Then if  $\alpha = a/E \in \widetilde{\mathfrak{C}}^{\text{eq}}$ , then  $\alpha \in \text{Cb}(a/\text{acl}^{\text{eq}}(\alpha)) \subseteq \text{acl}^{\text{eq}}(\alpha)$ .

□

### 3 Classification of models of $\widehat{T}$

In this section, we prove the main model-theoretic result of this paper, Theorem 3.31 below, which classifies the models of  $\widehat{T}$ .

#### 3.1 Outline

The classification proceeds as follows. First, recall the following coarse version of the classification of models of  $T$ . By [Las85, Theorem 6],  $T$  is almost  $\aleph_1$ -categorical. It follows ([Bue96, 7.1]) that if  $M \models T$  and  $M_0 \prec M$  is a copy of the prime model, there is a finite set of mutually orthogonal strongly minimal sets  $D_i$  defined over  $M_0$  such that  $M$  is constructible and minimal over  $M_0 B$ , where  $B$  is the union of arbitrary  $\text{acl}$ -bases over  $M_0$  for the  $D_i(M)$  ([Bue96, 7.1.2(ii)]).

We will show that this picture lifts to  $\widehat{T}$ . We will show that an arbitrary model  $\widetilde{M} \models \widehat{T}$  is constructible and minimal over  $\widetilde{M}_0 B$  where  $\widetilde{M}_0 = \rho^{-1}(M_0)$ , and  $M_0 \prec M$  and  $B$  are as above. So models of  $\widehat{T}$  are determined up to isomorphism by a choice of model of  $T$  and a choice of lift of the prime model  $M_0 \prec M$  (which in particular involves a choice of kernel).

In the case considered in the introduction, where  $\mathbb{G}$  is an algebraic group over  $k_0$ , we need just one strongly minimal set  $D$ , which we can take to be an algebraically closed field with parameters for  $k_0$ . Then  $M_0 \cong \mathbb{G}(k_0^{\text{alg}})$ , and for  $\mathbb{G}(K) \models T$ , the basis  $B$  is a transcendence base for  $K$  over  $k_0^{\text{alg}}$ .

#### 3.2 Preliminaries

We work in a monster model  $\widetilde{\mathfrak{C}} \models \widehat{T}$  and the corresponding monster model  $\mathfrak{C} = \rho(\widetilde{\mathfrak{C}}) \models T$ .

However, we mostly want to consider only those elementary embeddings of models of  $\widehat{T}$  which preserve the kernel.

**Notation 3.1.** For  $\widetilde{M} \subseteq \widetilde{N}$  models of  $\widehat{T}$ , we write  $\widetilde{M} \prec^* \widetilde{N}$  to mean that  $\widetilde{M} \prec \widetilde{N}$  and  $\ker(\widetilde{M}) = \ker(\widetilde{N})$ . We refer to such elementary embeddings as *kernel-preserving*.

*Remark 3.2.* If  $\widetilde{M} \prec \widetilde{N}$ , then  $\widetilde{M} \prec^* \widetilde{N}$  iff  $\widetilde{M} = \rho^{-1}(M) \subseteq \widetilde{N}$ , the inverse image of  $M$  evaluated in  $\widetilde{N}$ .

**Lemma 3.3.** *If  $\widetilde{N} \models \widehat{T}$  and  $M \prec N = \rho(\widetilde{N})$ , then  $\widetilde{M} := \rho^{-1}(M) \prec^* \widetilde{N}$ .*

*Proof.* In light of Remark 3.2 and the quantifier elimination, it suffices to show that  $\widetilde{M} \models \widehat{T}$ . For this, we check that the axioms (A1)-(A9) hold. These all follow straightforwardly from  $M$  being an elementary submodel of  $N$  and the kernel being preserved, except for the surjectivity in (A9) which is a little less straightforward. For that, with notation as in (D3) of Proposition 2.8, note that  $\rho_e(\widehat{H}(\widetilde{M})) = H(M) \subseteq \text{pr}(\rho_e(\widehat{G}(\widetilde{M}))) = \rho_e(\text{pr}(\widehat{G}(\widetilde{M})))$ , so

$$\begin{aligned} \widehat{H}(\widetilde{M}) &\subseteq \text{pr}(\widehat{G}(\widetilde{M})) + \ker(\rho_e|_{\widehat{H}(\widetilde{M})}) \\ &= \text{pr}(\widehat{G}(\widetilde{M})) + e(\widehat{H}(\widetilde{M}) \cap \ker) \\ &= \text{pr}(\widehat{G}(\widetilde{M})), \end{aligned}$$

using that (D3) holds for  $\widetilde{N}$ , and the kernel preservation.  $\square$

We make extensive use of l-isolation, a technique due to Lachlan [Lac73].

**Definition 3.4.** A type  $p$  is l-isolated if for each  $\phi(x, y)$  there exists  $\psi(x) \in p$  such that  $\psi$  implies the complete  $\phi$ -type implied by  $p$ ,  $\psi \models p|_{\phi}$ .

Recall that  $A$  is *atomic* over  $B$  if  $\text{tp}(a/B)$  is isolated for each tuple  $a \in A$ , and  $A$  is *constructible* over  $B$  if  $A$  has an enumeration  $(a_i)_{i < \lambda}$  such that  $\text{tp}(a_i/Ba_{<i})$  is isolated for each  $i < \lambda$ , where  $Ba_{<i} = B \cup \{a_j \mid i < j\}$ . We define *l-atomic* and *l-constructible* by replacing isolation with l-isolation in these definitions.

*Remark 3.5.* This definition of l-isolation is easily seen to be equivalent to the  $F_{\aleph_0}^l$ -isolation of [She90, Definition IV.2.3].

Clearly any isolated type is l-isolated, so atomicity implies l-atomicity and constructibility implies l-constructibility.

It is easy to see that, just as for constructibility and atomicity in their usual senses, l-constructibility implies l-atomicity ([She90, Theorem IV.3.2]), and the converse holds for countable sets ([She90, Lemma IV.3.16]).

**Lemma 3.6.** (a) *Work in a monster model  $\mathfrak{C}'$  of a complete stable theory.*

- (i) *l-constructible models exist over arbitrary sets: for  $A \subseteq \mathfrak{C}'^{\text{eq}}$ , there exists  $M \prec \mathfrak{C}'$  such that  $A \subseteq M^{\text{eq}}$  and  $M^{\text{eq}}$  is l-constructible over  $A$ .*
- (ii) *If  $M \prec \mathfrak{C}'$ , and  $\phi$  is a formula over  $M$  such that  $\phi(M) \subseteq A \subseteq \mathfrak{C}'^{\text{eq}}$  and  $\text{dcl}^{\text{eq}}(A) \cap \text{dcl}^{\text{eq}}(\phi(\mathfrak{C}')) \subseteq M^{\text{eq}}$ , and if  $b$  is l-isolated over  $A$  and  $\models \phi(b)$ , then  $b \in \phi(M)$ .*

(b) *If  $\widetilde{M} \models \widehat{T}$  and  $\rho(\widetilde{M}) =: M \prec N \models T$ , and  $\widetilde{N} \models \widehat{T}$  is l-atomic over  $A := \widetilde{M} \cup N$ , then  $\widetilde{N} \prec^* \widetilde{M}$  and  $\rho(\widetilde{N}) = N$ , and so  $\widetilde{N}$  is minimal over  $A$ .*

*Proof.* (a)

- (i) [She90, IV.2.18(4), IV.3.1(5)]
- (ii) If  $b \notin \phi(M)$ , then by l-isolation, there is a formula  $\psi \in \text{tp}(b/A)$  such that

$$\psi(x) \models \phi(x) \wedge x \notin \phi(M).$$

By stable embeddedness of  $\phi$ , we may take  $\psi$  to be defined over  $\text{dcl}^{\text{eq}}(A) \cap \text{dcl}^{\text{eq}}(\phi(\mathfrak{C}')) \subseteq M^{\text{eq}}$ . But then  $\psi$  is realised in  $M$ , which is a contradiction.

- (b) This follows from (a)(ii) and the QE. Indeed, if  $\beta \in \text{dcl}^{\text{eq}}(\widetilde{\zeta})$  with  $\widetilde{\zeta}$  a tuple from  $\ker(\widetilde{\mathfrak{C}})$ , then since  $\rho_n(\widetilde{\zeta}) \in \text{Tor} \subseteq M$ , the QE implies that  $\text{tp}(\widetilde{\zeta}/A)$  is determined by  $\text{tp}(\widetilde{\zeta}/\widetilde{M})$ . So if  $\beta \in \text{dcl}^{\text{eq}}(A)$ , then already  $\beta \in \widetilde{M}^{\text{eq}}$ . So by (a)(ii),  $\ker(\widetilde{N}) = \ker(\widetilde{M})$ .

Similarly, if  $\beta \in \text{dcl}^{\text{eq}}(b)$  with  $b$  a tuple from  $\mathfrak{C}$ , then the QE implies that  $\text{tp}(b/A)$  is determined by  $\text{tp}(b/N)$ . Let  $\widehat{N} \models \widehat{T}$  be the profinite universal cover, embedded in  $\widetilde{\mathfrak{C}}$  over  $N$ . Then  $\text{tp}(b/A)$  is determined by  $\text{tp}(b/\widehat{N})$ , again, if  $\beta \in \text{dcl}^{\text{eq}}(A)$ , then already  $\beta \in \widehat{N}^{\text{eq}}$ . So by (a)(ii), we have  $\rho(\widehat{N}) = \rho(\widetilde{N}) = N$ .

The claimed minimality is then clear, since  $\rho$  is a homomorphism.  $\square$

We also use the existence of l-constructible models to obtain independent amalgamation in the (abstract elementary) class  $(\text{Mod}(\widehat{T}), \prec^*)$  of models of  $\widehat{T}$  with kernel-preserving embeddings.

**Lemma 3.7.** *Suppose  $\widetilde{M}_i$ ,  $i = 0, 1, 2$ , are elementary submodels of  $\widetilde{\mathfrak{C}}$ ,  $\widetilde{M}_0 \prec^* \widetilde{M}_i$ , and  $\widetilde{M}_1 \downarrow_{\widetilde{M}_0} \widetilde{M}_2$ . Let  $\widetilde{M}_3$  be an l-atomic model over  $\widetilde{M}_1 \cup \widetilde{M}_2$ . Then  $\widetilde{M}_i \prec^* \widetilde{M}_3$ .*

*Proof.* Suppose  $\widetilde{\zeta} \in \ker(\widetilde{M}_3) \setminus \ker(\widetilde{M}_0)$ . By l-atomicity, say  $\phi(x, \widetilde{a}_1, \widetilde{a}_2) \in \text{tp}(\widetilde{\zeta}/\widetilde{M}_1 \cup \widetilde{M}_2)$  with  $a_i \in \widetilde{M}_i$  and

$$\phi(x, \widetilde{a}_1, \widetilde{a}_2) \vdash x \notin \ker(\widetilde{M}_0) \wedge x \in \ker.$$

By Corollary 2.12, we may assume  $\phi(x, \widetilde{a}_1, \widetilde{a}_2)$  is of the form  $x \in \widehat{H}(\widetilde{a}_1, \widetilde{a}_2) \wedge \rho_n(x) = \zeta_n$ , where  $\zeta_n = \rho_n(\widetilde{\zeta}) \in \widetilde{M}_0$ .

By the independence,  $\text{tp}(\widetilde{a}_1/\widetilde{M}_2)$  is finitely satisfiable in  $\widetilde{M}_0$ , so say  $\widetilde{a}'_1 \in \widetilde{M}_0$  and  $\widetilde{M}_2 \models \exists x \in \ker \cdot \phi(x, \widetilde{a}'_1, \widetilde{a}_2)$ , witnessed say by  $\widetilde{\zeta}' \in \ker(\widetilde{M}_2) = \ker(\widetilde{M}_0)$ .

Then  $\text{tp}(\widetilde{\zeta} - \widetilde{\zeta}'/\widetilde{M}_1) \ni (x \in \widehat{H}(\widetilde{a}_1 - \widetilde{a}'_1, 0) \wedge \rho_n(x) = 0)$ , so say  $\widetilde{\zeta}'' \in \widetilde{M}_1$  also satisfies this. Then  $\widetilde{\zeta}' + \widetilde{\zeta}'' \in \widehat{H}(\widetilde{a}'_1 + \widetilde{a}_1 - \widetilde{a}'_1, \widetilde{a}_2 + 0) = \widehat{H}(\widetilde{a}_1, \widetilde{a}_2)$  and  $\rho_n(\widetilde{\zeta}' + \widetilde{\zeta}'') = \zeta_n + 0 = \zeta_n$ ; but  $\widetilde{\zeta}' + \widetilde{\zeta}'' \in \ker(\widetilde{M}_1) = \ker(\widetilde{M}_0)$ , contradicting the choice of  $\phi$ .  $\square$

**Lemma 3.8.** *Suppose  $\widetilde{M} \models \widehat{T}$  and  $A \subseteq \widetilde{M}^{\text{eq}}$  with  $\ker(\widetilde{M}) \subseteq A$ , suppose  $M$  is countable, and suppose  $\widetilde{M}$  is atomic over  $A$ . Then  $\widetilde{M}$  is constructible over  $A$ .*

*Proof.* Take an arbitrary section  $S \subseteq \widetilde{M}$  of  $\rho : \widetilde{M} \rightarrow M$ . Then  $S$  is countable and atomic, and hence constructible, over  $A$ , and  $\widetilde{M} = S + \ker$  is clearly constructible over  $S \cup A \supseteq S \cup \ker$ .  $\square$

**Lemma 3.9.** *Suppose  $\widetilde{B} \subseteq \widetilde{M} \models \widehat{T}$  and  $\widetilde{a} \in \widetilde{M}$ , and each  $\text{tp}(a_m/\widetilde{B})$  is isolated. Then  $\text{tp}(\widetilde{a}/\widetilde{B})$  is l-isolated.*

*Proof.* By the QE, it suffices to see that  $\text{tp}_\phi(\widetilde{a}/\widetilde{B})$  is isolated for an atomic formula  $\phi(x, y)$ . For  $\phi$  of the form  $\psi(\rho_m(x), \rho_m(y))$ , this follows from  $\text{tp}(a_m/\widetilde{B})$  being isolated. For  $\phi$  of the form  $(x, y) \in \widehat{H}$ , it follows from the fact that for  $\widetilde{b} \in \widetilde{B}$ ,  $(\widetilde{a}, \widetilde{b}) \in \widehat{H} \Leftrightarrow \text{grploc}(\widetilde{a}/\widetilde{B}) \subseteq \widehat{H}(\widetilde{b})$ .  $\square$

### 3.3 $\omega$ -stability over models

From now on, in order to prove the subsequent lemma, we make the following additional assumption.

**Assumption 3.10.**  $T$  is *rigid* - for  $\mathbb{G}$  a saturated model of  $T$ , every connected definable subgroup  $H$  of  $\mathbb{G}^n$  is defined over  $\text{acl}^{\text{eq}}(\emptyset)$  - and hence, by our previous assumptions in Subsection 2.2, is actually defined over  $\emptyset$ . So  $\widehat{L}$  has a predicate  $\widehat{H}$  corresponding to  $H$ .

We now apply the “Kummer theory over models” of [BGH14] to obtain atomicity of “finitely generated” extensions of models.

**Lemma 3.11.** *Suppose  $\widetilde{M} \prec \mathfrak{C}$  and  $b \in \mathfrak{C}$ , and let  $M(b)$  be a prime model over  $Mb$ . Suppose  $\widetilde{M}(b)$  is a model such that  $\widetilde{M} \prec^* \widetilde{M}(b) \prec \mathfrak{C}$  and  $\rho(\widetilde{M}(b)) = M(b)$ . Then  $\widetilde{M}(b)$  is atomic over  $\widetilde{M}b$ . If  $M$  is countable,  $\widetilde{M}(b)$  is constructible over  $\widetilde{M}b$ .*

*Furthermore, such an  $\widetilde{M}(b)$  exists.*

*Proof.* We first show the atomicity. Let  $\tilde{c} \in \widetilde{M}(b)$ ; we must show that  $\text{tp}(\tilde{c}/\widetilde{M}b)$  is isolated. Let  $\widehat{H} + \tilde{d} = \text{grploc}(\tilde{c}/\widetilde{M})$ . Since  $\widetilde{M}$  is a model, we may assume  $\tilde{d} \in \widetilde{M}$ . So by replacing  $\tilde{c}$  with  $\tilde{c} - \tilde{d}$ , we may assume  $\tilde{d} = 0$ .

Since  $\widetilde{M}$  contains  $\ker(\widetilde{M}(b))$  and  $T$  is rigid,  $c$  is *free* in  $H$  over  $M$ , i.e. in no proper coset defined over  $M$ . By [BGH14, 6.4], for some  $n$ , writing  $\widehat{x}$  for the long tuple of variables  $(x_i)_{i>0}$ ,

$$\text{tp}(c_n/M)(x_n) \cup \{x_i \in H \mid i > 0\} \models \text{tp}(\widehat{c}/M)(\widehat{x}).$$

Now by  $\omega$ -stability of  $T$ ,  $\text{tp}(b/Mc)$  has finite multiplicity, i.e. finitely many extensions to  $\text{acl}^{\text{eq}}(Mc) \supseteq \widehat{c}$ . Hence  $\text{tp}(\widehat{c}/M) \cup \text{tp}(c/Mb)$  has only finitely many extensions to  $Mb$ . So again, for some  $n$ ,

$$\text{tp}(c_n/Mb)(x_n) \cup \{x_i \in H \mid i > 0\} \models \text{tp}(\widehat{c}/Mb)(\widehat{x}).$$

So by Lemma 2.16,

$$\text{tp}(c_n/Mb)(\rho_n(\tilde{x})) \cup \{\tilde{x} \in \widehat{H}\} \models \text{tp}(\tilde{c}/\widetilde{M}b)(\tilde{x}).$$

But  $c_n \in M(b)$ , so  $\text{tp}(c_n/Mb)$  is isolated, so  $\text{tp}(\tilde{c}/\widetilde{M}b)$  is isolated.

This proves atomicity. Constructibility assuming countability of  $M$  follows by Lemma 3.8.

It remains to show existence. By Lemma 3.6(a)(i), there exists a model  $\widetilde{M}(b)$  which is l-constructible over  $\widetilde{M} \cup M(b)$ , and by Lemma 3.6(b) the kernel is preserved and  $\rho(\widetilde{M}(b)) = M(b)$ . □

*Remark 3.12.* Note that  $\widetilde{M}(b)$  will **not** be constructible over  $\widetilde{M} \cup M(b)$ : indeed, if  $\tilde{a} \in \widetilde{M}(b) \setminus \widetilde{M}$ , then each  $a_n$  is in  $M(b) \setminus M$ , so easily  $\text{tp}(\tilde{a}/\widetilde{M} \cup M(b))$  is not isolated.

*Remark 3.13.* If we don’t assume rigidity, there could be subgroups definable over  $M(b)$  which aren’t definable over  $M$ , which could cause a failure of atomicity.



*Remark 3.14.* Lemma 3.11 implies that we have  $\omega$ -stability over models in the (abstract elementary) class  $(\text{Mod}(\widehat{T}), \prec^*)$ , in the sense that if  $\widetilde{M} \models \widehat{T}$  is countable, then there are only countably many types over  $\widetilde{M}$  realised in kernel-preserving extensions of  $\widetilde{M}$ . Indeed, by Lemma 3.11 any such type is isolated over  $\widetilde{M}b$  for some  $b$ , and by  $\omega$ -stability of  $T$  there are only countably many possible types  $\text{tp}(b/\widetilde{M}) \equiv \text{tp}(b/M)$ . We will see in Remark 3.32 below that the Galois type of  $b$  over  $\widetilde{M}$  is determined by  $\text{tp}(b/\widetilde{M})$ , which means that we have  $\omega$ -stability over models in the sense of the abstract elementary class.

### 3.4 Independent systems

Countability of  $M$  was crucial to get constructibility in Lemma 3.11. For constructibility of extensions in higher cardinals, we require constructibility over independent systems of models. [She90, XII] and [Har87] are the sources for the techniques used here.

In this subsection, we develop what we need of the general theory of independent systems. We work in a monster model  $\mathfrak{C}'$  of an arbitrary stable theory  $T'$ .

**Definition 3.15.** If  $I$  is a downward-closed set of sets, an *I-system* in  $\mathfrak{C}'$  is a collection  $(M_s \mid s \in I)$  of elementary submodels  $\mathfrak{C}'$  such that for  $s \subseteq t$ ,  $M_s$  is an elementary submodel of  $M_t$ . For  $J \subseteq I$ , define  $M_J := \bigcup_{s \in J} M_s \subseteq \mathfrak{C}'$ .

Define  $\prec s := \mathcal{P}^-(s) := \mathcal{P}(s) \setminus \{s\}$ , and  $\not\prec s := I \setminus \{t \mid t \supseteq s\}$ .

The system is *constructible* if  $M_s$  is constructible over  $M_{\prec s}$  for all  $s \in I$  with  $|s| > 1$ . Similarly for *atomic*, and for *l-constructible* and *l-atomic*.

The system is *independent* (or *stable*) if  $M_s \downarrow_{M_{\prec s}} M_{\not\prec s}$  for all  $s \in I$ .

$I$  is *Noetherian* if each  $s \in I$  is finite.

An *enumeration* of  $I$  is a sequence  $(s_i)_{i \in \lambda}$  such that  $I = \{s_i \mid i \in \lambda\}$  and  $s_i \subseteq s_j \Rightarrow i \leq j$ . We then write  $s_{< i}$  for  $\{s_j \mid j < i\}$ .

We define  $|n| := \{0, \dots, n-1\}$ .

Note that if  $(s_i)_{i \in \lambda}$  is an enumeration of an independent  $I$ -system, then we have  $M_{s_i} \downarrow_{M_{s_{< i}}} M_{s_{< i}}$  for all  $i$ . That the converse holds is given by the following Fact, which is [She90, Lemma XII.2.3(1)].

**Fact 3.16.** *Let  $(M_s)_s$  be an  $I$ -system, let  $(s_i)_{i \in \lambda}$  be an enumeration, and suppose  $M_{s_i} \downarrow_{M_{s_{< i}}} M_{s_{< i}}$  holds for all  $i$ . Then the system is independent.*

**Definition 3.17.** Let  $M$  be a (possibly multi-sorted) structure. If  $A \subseteq B \subseteq M$ , we say  $A$  is *Tarski-Vaught* in  $B$ ,  $A \subseteq_{TV} B$ , if every formula over  $A$  which is realised in  $B$  is realised in  $A$ .

**Lemma 3.18.** *Suppose  $C \subseteq_{TV} B \subseteq M$ .*

(i) *If a type  $\text{tp}(a/C)$  is  $l$ -isolated, then  $\text{tp}(a/C) \models \text{tp}(a/B)$ , and  $Ca \subseteq_{TV} Ba$ .*

(ii) *If  $A \subseteq M$  is constructible over  $C$  then  $A$  is constructible over  $B$ .*

(iii) *If  $A \subseteq M$  is  $l$ -atomic over  $C$ , then  $A \downarrow_C B$ .*

*Proof.* (i) Given  $\phi(x, y)$ , say  $\psi(x) \in \text{tp}(a/C)$  isolates  $\text{tp}_\phi(a/C)$ . Then for  $b \in B$ ,  $\phi(x, b) \in \text{tp}_\phi(a/B)$  iff  $\psi(x) \models \phi(x, b)$ ; indeed, else

$$\models (\exists x. \psi(x) \wedge \phi(x, b)) \wedge (\exists x. \psi(x) \wedge \neg \phi(x, b));$$

but then the same holds for some  $c \in C$ , contradicting the isolation.

So  $\text{tp}(a/C) \models \text{tp}(a/B)$ . Also  $Ca \subseteq_{TV} Ba$ , since if  $\models \phi(a, b)$  then  $\models \forall x. \psi(x) \rightarrow \phi(x, b)$ , hence this holds for some  $c \in C$ , and hence  $\models \phi(a, c)$ .

(ii) This follows from (i) by a transfinite induction.

(iii) The extension of  $\text{tp}(A/C)$  to  $B$  is unique by (i), so must be the non-forking extension. □

**Lemma 3.19.** *Suppose  $M$  is a model, and  $A \downarrow_M B$ . Then  $MA \subseteq_{TV} MAB$ .*

*Proof.* By the coheir property of non-forking over models in stable theories,  $\text{tp}(B/MA)$  is finitely satisfiable in  $M$ . □

The following is [She90, Lemma XII.2.3(2)], to which we refer for the proof.

**Fact 3.20 (TV Lemma).** *If  $(M_s)_s$  is an independent  $I$ -system in a stable theory, if  $J \subseteq I$ , and if  $\forall s \in I. (s \subseteq \bigcup J \Rightarrow s \in J)$ , then  $M_J \subseteq_{TV} M_I$ .*

**Lemma 3.21.** *Let  $(M_s)_s$  be a constructible Noetherian independent  $I$ -system. Suppose that for each  $p \in \bigcup I$ ,  $B_p$  is a subset of  $M_{\{p\}}$  for which  $M_{\{p\}}$  is constructible over  $M_\emptyset B_p$ .*

*Then  $M_I$  is constructible over  $M_\emptyset \cup \bigcup_{p \in \bigcup I} B_p =: A$ .*

*Proof.* Let  $(s_i)_{i < \lambda}$  be an enumeration of  $I$ . It suffices to show that each  $M_{s_i}$  is constructible over  $AM_{s_{<i}}$ , as it then follows by induction on  $i \leq \lambda$  that  $M_{s_{<i}}$  is constructible over  $A$ .

If  $s_i = \emptyset$ , the constructibility is immediate. If  $s_i = \{p\}$ , we have  $B_p \downarrow_{M_\emptyset} (M_{s_{<i}} \cup (A \setminus B_p))$ . So by Lemma 3.19,  $M_\emptyset B_p \subseteq_{TV} AM_{s_{<i}}$ . The desired constructibility then follows from Lemma 3.18(ii).

If  $|s_i| > 1$ , then  $M_{s_i}$  is constructible over  $M_{s_{<i}}$ ; but  $M_{s_{<i}} \subseteq_{TV} M_{s_{<i} \cup \{p\} \mid p \in \bigcup I}$  by the TV Lemma, so in particular  $M_{s_{<i}} \subseteq_{TV} AM_{s_{<i}}$ . Again, Lemma 3.18(ii) yields the desired constructibility. □

**Lemma 3.22.** *Suppose  $\bigcup I$  is finite.*

*For an  $l$ -atomic  $I$ -system to be independent, it suffices that for each  $p \in \bigcup I$ ,*

$$M_{\{p\}} \downarrow_{M_\emptyset} M_{\neq \{p\}}.$$

*Proof.* Suppose inductively that for any downward closed proper subset  $J$  of  $I$ , the restriction of the  $I$ -system to a  $J$ -system is independent.

So it suffices to show that for  $s \in I$  maximal,  $M_s \downarrow_{M_{<s}} M_{I \setminus \{s\}}$ .

If  $|s| = 1$ , this holds by assumption.

If  $|s| > 1$ , then if  $t \subseteq s$  and  $t \in I \setminus \{s\}$  then  $t \in <s$ , so by the TV Lemma applied to the restricted independent  $(I \setminus \{s\})$ -system,

$$M_{<s} \subseteq_{TV} M_{I \setminus \{s\}}.$$

But  $M_s$  is  $l$ -atomic over  $M_{<s}$ , so we conclude the independence by Lemma 3.18(iii). □

### 3.5 Atomicity over independent systems in $\widehat{T}$

Now we return to considering  $\widehat{T}$  and  $T$ .

Let  $M \models T$ , and let  $M_0 \prec M$  be a copy of the prime model, and let  $D_i$  and  $B_i$  be as in Subsection 3.1. Let  $B := \bigcup_i B_i$ , and let  $\mathcal{P}^{\text{fin}}(B)$  be the set of finite subsets of  $B$ . Let  $M_\emptyset = M_0$ , and for  $s \in \mathcal{P}^{\text{fin}}(B)$  inductively let  $M_s \prec M$  be prime over  $M_{\prec s} \cup s$ .

**Lemma 3.23.**  $(M_s)_{s \in \mathcal{P}^{\text{fin}}(B)}$  is a constructible independent  $\mathcal{P}^{\text{fin}}(B)$ -system, and  $\bigcup_s M_s = M$ .

*Proof.*  $\bigcup_s M_s$  is an elementary submodel of  $M$  which contains  $M_0 B$ , and  $M$  is minimal over  $M_0 B$ , so  $\bigcup_s M_s = M$ .

The system is constructible by construction, prime models being constructible in  $\omega$ -stable theories. For independence, by finite character of forking and Lemma 3.22 it suffices to see that  $M_{\{b\}} \downarrow_{M_0} M_s$  when  $b \notin s \in \mathcal{P}^{\text{fin}}(B)$ .

We may assume inductively that the restriction of the system to  $s$  is independent. So by Lemma 3.21,  $M_s$  is constructible over  $M_0 s$ .

Now  $b \notin M_s$  since (by orthogonality of the  $D_i$ )  $\text{tp}(b/M_0 s)$  is not algebraic and hence not isolated.

So  $bM_0 \downarrow_{M_0} M_s$ . So by Lemma 3.19,  $bM_0 \subseteq_{TV} bM_s$ . Since  $M_{\{b\}}$  is constructible and hence atomic over  $bM_0$ , it follows by Lemma 3.18(iii) that  $M_{\{b\}} \downarrow_{bM_0} bM_s$ , and in particular  $M_{\{b\}} \downarrow_{bM_0} M_s$ . So by transitivity,  $M_{\{b\}} \downarrow_{M_0} M_s$ .  $\square$

**Definition 3.24.** An  $I$ - $\sim$ -system is an  $I$ -system  $(\widetilde{M}_s)_s$  in  $\widetilde{\mathcal{C}} \models \widehat{T}$  such that

- setting  $M_s := \rho(\widetilde{M}_s) \models T$ ,  $(M_s)_s$  is an independent atomic  $I$ -system in  $T$ ;
- $\widetilde{M}_s \prec^* \widetilde{M}_t$  when  $s \subseteq t$ .

The definition assumes only independence in  $T$ , but independence in  $\widehat{T}$  follows:

**Lemma 3.25.** An  $I$ - $\sim$ -system  $(\widetilde{M}_s)_s$  is an independent  $I$ -system.

*Proof.* Let  $(s_i)_{i \in \lambda}$  be an enumeration of  $I$ . By Fact 3.16, it suffices to show that given  $i \in \lambda$ , we have  $\widetilde{M}_{s_i} \downarrow_{\widetilde{M}_{\prec s_i}} \widetilde{M}_{s_{\prec i}}$ , where we may assume inductively that the restriction of  $(\widetilde{M}_s)_s$  to  $s_{\prec i}$  is an independent system.

By Proposition 2.21(ii) and the independence of  $(M_s)_s$ , it suffices to show that for  $\widetilde{a} \in \widetilde{M}_{s_i}$ , we have  $C := \text{grploc}(\widetilde{a}/\widetilde{M}_{s_{\prec i}})$  is defined over  $\widetilde{M}_{\prec s_i}$ . Say  $C = \widehat{H}(\widetilde{b}')$  with  $\widetilde{b}' \in \widetilde{M}_{s_{\prec i}}$ .

Now  $aM_{\prec s_i} \subseteq_{TV} aM_{s_{\prec i}}$ , by the TV Lemma (Fact 3.20) and Lemma 3.18(i) if  $|s_i| > 1$ , and by Lemma 3.19 if  $|s_i| = 1$ .

So  $H(\widetilde{b}') = H(0) + a = H(b)$  for some  $b \in M_{\prec s_i}$ . So say  $\widetilde{b} \in \widetilde{M}_{\prec s_i}$  and  $\rho(\widetilde{b}) = b$ ; then  $\widetilde{a} + \zeta \in \widehat{H}(\widetilde{b})$  for some  $\zeta \in \ker(\widetilde{M}_{s_i}) = \ker(\widetilde{M}_\emptyset)$ . So  $\widehat{H}(\widetilde{b}') = \widehat{H}(\widetilde{b}) - \zeta$ , which (by Remark 2.15) is defined over  $\widetilde{M}_{\prec s_i}$ .  $\square$

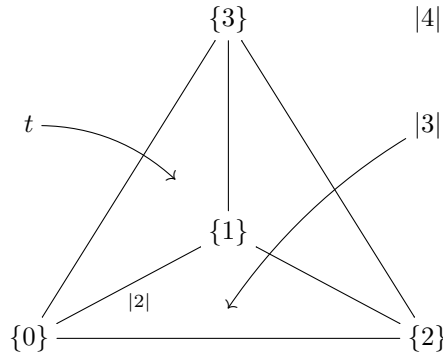
**Proposition 3.26.** Let  $(\widetilde{M}_s)_s$  be an  $I$ - $\sim$ -system with  $I$  Noetherian. Then the system is atomic. If also each  $M_s$  is countable, then the system is constructible.

*Proof.* It suffices to show this for  $I = \mathcal{P}(|n|)$ ,  $n > 1$ , where recall  $|n| \models \{0, \dots, n-1\}$ . Indeed, by Noetherianity, the system below any  $s \in I$  is of this form. We inductively assume the result for  $1 < n' < n$ .

We show that  $\widetilde{M}_{|n|}$  is atomic over  $\widetilde{M}_{<|n|}$ . Constructibility assuming countability then follows by Lemma 3.8.

**Claim 3.27.**  $(\widetilde{M}_s)_s$  extends to a  $\mathcal{P}(|n+1|)$ - $\sim$ -system such that  $\widetilde{M}_{|n|}$  is isomorphic over  $\widetilde{M}_{|n-1|}$  to  $\widetilde{M}_{|n-1| \cup \{n\}}$ , by an isomorphism  $\sigma$  such that moreover  $\sigma(\widetilde{M}_s) = \widetilde{M}_{(s \setminus \{n-1\}) \cup \{n\}}$  for  $s \subseteq |n|$ .

*Proof.* Let  $t := |n-1| \cup \{n\}$ .



Let  $\widetilde{M}_t$  be a realisation of  $\text{tp}(\widetilde{M}_{|n|}/\widetilde{M}_{|n-1|})$  independent from  $\widetilde{M}_{|n|}$ , and let  $\sigma : \widetilde{M}_{|n|} \xrightarrow{\cong} \widetilde{M}_{|n-1|} \widetilde{M}_t$  be an isomorphism witnessing the equality of types. Let  $\widetilde{\mathcal{M}}$  be an  $\mathcal{l}$ -atomic model over  $\widetilde{M}_{|n|} \cup \widetilde{M}_t$ . By Lemma 3.7,  $\ker(\widetilde{\mathcal{M}}) = \ker(M_\emptyset)$ .

We define an enumeration  $s_i$  of  $\mathcal{P}(|n+1|)$ , and recursively define  $\widetilde{M}_{s_i} \prec^* \widetilde{\mathcal{M}}$  such that

$$M_{s_i} \downarrow_{M_{<s_i}} M_{s_{<i}}$$

and  $M_{s_i}$  is atomic over  $M_{<s_i}$ . By Fact 3.16, this will yield a  $\mathcal{P}(|n+1|)$ - $\sim$ -system.

Begin with an enumeration of  $\mathcal{P}(|n|)$ ; the corresponding  $\widetilde{M}_{s_i} \prec^* \widetilde{\mathcal{M}}$  are already given.

Continue with an enumeration of  $\mathcal{P}(t)$ , setting  $\widetilde{M}_{s_i} := \sigma(\widetilde{M}_{(s_i \setminus \{n\}) \cup \{n-1\}}) \prec^* \widetilde{M}_t \prec^* \widetilde{\mathcal{M}}$ . For the independence condition, we have  $M_{s_i} \downarrow_{M_{<s_i}} M_{s_{<i} \cap \mathcal{P}(t)}$  since  $s_i$  is part of an enumeration of  $\mathcal{P}(t)$ , and then by transitivity and  $M_t \downarrow_{M_{|n-1|}} M_{|n|}$  we deduce  $M_{s_i} \downarrow_{M_{<s_i}} M_{s_{<i} \cap \mathcal{P}(t)} M_{|n|}$  and hence  $M_{s_i} \downarrow_{M_{<s_i}} M_{s_{<i}}$ .

Now for the remaining  $s_i$ : let  $M'_{s_i} \prec \mathcal{M}$  be a constructible model over  $M_{<s_i} \subseteq \mathcal{M}$ , and let  $\widetilde{M}_{s_i}$  be the inverse image in  $\widetilde{\mathcal{M}}$ . The TV Lemma (Fact 3.20) gives  $M_{<s_i} \subseteq_{TV} M_{s_{<i}}$ , so  $M_{s_i} \downarrow_{M_{<s_i}} M_{s_{<i}}$  by Lemma 3.18(iii).  $\square$

Define

$$\begin{aligned}
\widetilde{\Delta} &:= \widetilde{M}_{|n|} & \widetilde{\Delta}' &:= \widetilde{M}_{|n+1|} \\
d_i \widetilde{\Delta} &:= \widetilde{M}_{|n| \setminus \{i-1\}} & d_i \widetilde{\Delta}' &:= \widetilde{M}_{|n+1| \setminus \{i-1\}} \\
d \widetilde{\Delta} &:= \bigcup_{1 \leq i \leq n} d_i \widetilde{\Delta} & d \widetilde{\Delta}' &:= \bigcup_{1 \leq i \leq n} d_i \widetilde{\Delta}' \\
\widetilde{\Lambda} &:= \bigcup_{1 \leq i < n} d_i \widetilde{\Delta} & \widetilde{\Lambda}' &:= \bigcup_{1 \leq i < n} d_i \widetilde{\Delta}' \\
d d_i \widetilde{\Delta} &:= \bigcup_{j \in |n| \setminus \{i-1\}} \widetilde{M}_{|n| \setminus \{i-1, j\}}
\end{aligned}$$

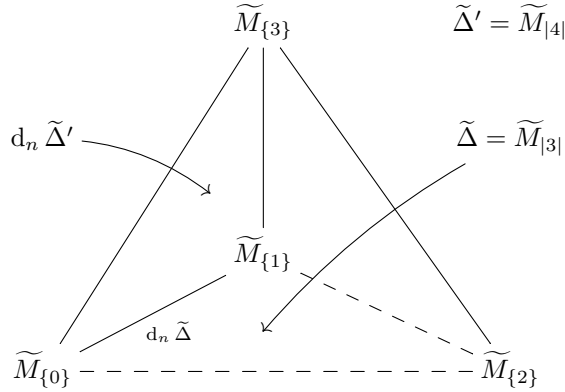
We also define the corresponding sets in  $T$ , e.g.  $\Lambda := \rho(\widetilde{\Lambda}) = \bigcup_{i < n-1} M_{|n| \setminus \{i\}}$ . In this notation, the isomorphism of the previous claim is

$$\sigma : \widetilde{\Delta} \xrightarrow{\cong}_{d_n \widetilde{\Delta}} d_n \widetilde{\Delta}'.$$

Note that it induces an isomorphism

$$\sigma : \Delta \xrightarrow{\cong}_{d_n \Delta} d_n \Delta'.$$

A diagram for  $n = 3$ :



the dashed lines indicate  $\widetilde{\Lambda}$ , and the faces above them form  $\widetilde{\Lambda}'$ .

Let  $\tilde{a} \in \widetilde{\Delta}$  be a tuple; we want to show that  $\text{tp}(\tilde{a}/d \widetilde{\Delta})$  is isolated.

**Claim 3.28.** *There exists  $b_0 \in d_n \Delta$  such that, setting  $A := \text{acl}^{\text{eq}}(d d_n \Delta b_0)$ ,*

$$\text{tp}(\hat{a}/A\Lambda) \vDash \text{tp}(\hat{a}/d \Delta).$$

*Proof.* Let  $b_0 \in d_n \Delta$  such that  $\text{tp}(a/d \Delta) \vDash \text{tp}(a/b_0 \Lambda)$ . First note that every extension of  $\text{tp}(a_m/b_0 \Lambda)$  to  $d \Delta$  is a non-forking extension. Indeed, that holds for  $m = 1$  by the uniqueness of the extension, and hence for any  $m$  by interalgebraicity of  $a_m$  with  $a$ . So it suffices to see that  $\text{tp}(a_m/A\Lambda)$  has a unique non-forking extension to  $d \Delta$ . So suppose  $c_1, c_2$  realise two such extensions. Then  $d_n \Delta \perp_{A\Lambda} c_i$ . Now  $\text{tp}(d_n \Delta/A)$  is stationary, and since the system is independent we have  $d_n \Delta \perp_{d d_n \Delta} \Lambda$  and hence  $d_n \Delta \perp_A A\Lambda$ , also  $\text{tp}(d_n \Delta/A\Lambda)$  is stationary. So  $c_1 \equiv_{d_n \Delta A\Lambda} c_2$ , so in particular  $c_1 \equiv_{d \Delta} c_2$ ,  $\square$

**Claim 3.29.**

$$\text{tp}(\widehat{a}/\sigma(\widehat{a})\Lambda'b_0) \models \text{tp}(\widehat{a}/A\Lambda)$$

*Proof.* Say  $\models \phi(a_n, b, e)$  where  $b \in A$  and  $e \in \Lambda$ .

Say  $\theta$  is an algebraic formula isolating  $\text{tp}(b/\text{d d}_n \Delta b_0)$ .

Let

$$\psi(x) := \forall y \in \theta. (\phi(x, y, e) \Leftrightarrow \phi(\sigma a_n, y, \sigma e)),$$

which is a formula over  $\sigma(a_n)\Lambda'b_0$  since  $\sigma e \in \sigma\Lambda \subseteq \Lambda'$ .

Then  $\psi(x) \models \phi(x, b, e)$ , since  $\models \phi(\sigma a_n, b, \sigma e)$ , since  $b \in \text{d}_n \Delta$  and  $\sigma : \Delta \xrightarrow{\cong} \text{d}_n \Delta$ , and similarly  $\psi(x) \in \text{tp}(a_n/\sigma(a_n)\Lambda'b_0)$ . So  $\text{tp}(\widehat{a}/\sigma(\widehat{a})\Lambda'b_0) \models \phi(a_n, b, e)$ .  $\square$

Now  $\text{d } \widetilde{\Delta} \subseteq_{TV} \text{d } \widetilde{\Delta}'$  by the TV lemma, and  $\text{tp}(\widetilde{a}/\text{d } \widetilde{\Delta})$  is l-isolated by Lemma 3.9, so by Lemma 3.18(i),  $\text{tp}(\widetilde{a}/\text{d } \widetilde{\Delta}) \models \text{tp}(\widetilde{a}/\text{d } \widetilde{\Delta}')$ .

Let  $\widetilde{b}_0 \in \rho^{-1}(b_0) \subseteq \text{d}_n \widetilde{\Delta}$ , and let  $\widetilde{b}_0 \subseteq \widetilde{b}'_0 \in \text{d}_n \widetilde{\Delta}$  be such that  $\text{grploc}(\widetilde{a}/\text{d } \widetilde{\Delta})$  is defined over  $\widetilde{b}'_0 \widetilde{\Lambda}$ . Then by Lemma 2.16 and the above Claims, we have:

$$\text{tp}(\widetilde{a}/\text{d } \widetilde{\Delta}) \models \text{tp}(\widetilde{a}/\sigma(\widetilde{a})\widetilde{\Lambda}'\widetilde{b}'_0).$$

So it suffices to see that the latter type is isolated.

If  $n > 2$ , we have that  $\text{tp}(\widetilde{a}\sigma(\widetilde{a})\widetilde{b}'_0/\widetilde{\Lambda}')$  is isolated by the inductive hypothesis applied to the  $\mathcal{P}(|n-1|)\text{-}\sim$ -system  $(\widetilde{M}'_s)_s$  defined by  $\widetilde{M}'_s := \widetilde{M}_{s \cup \{n-1, n\}}$ , since  $\widetilde{\Lambda}' = \widetilde{M}'_{<|n-1|}$  and  $\widetilde{M}'_{|n-1|} = \widetilde{M}_{|n+1|} = \widetilde{\Delta}'$ .

Finally, if  $n = 2$ , we claim that it follows from Lemma 3.11 that  $\text{tp}(\widetilde{a}\sigma(\widetilde{a})\widetilde{b}'_0/\widetilde{\Lambda}'b'_0)$  is isolated. Indeed,  $\widetilde{\Lambda}' = \widetilde{M}_{\{1,2\}}$ , and so it suffices to show that  $\text{tp}(a, \sigma(a)/M_{\{1,2\}}b'_0)$  is isolated, since then for an appropriate embedding of the prime model  $M_{\{1,2\}}(b'_0)$  into  $\Delta'$ , we have  $a, \sigma(a) \in M_{\{1,2\}}(b'_0)$ .

We conclude by proving this isolation of  $\text{tp}(a, \sigma(a)/M_{\{1,2\}}b'_0)$ . By the definitions of  $b_0$  and  $b'_0$ , we have that  $\text{tp}(a/b'_0 M_{\{1\}})$  implies  $\text{tp}(a/M_{\{0\}} M_{\{1\}})$  and so is isolated, and hence by  $M_{\{0,1\}} \downarrow_{M_{\{1\}}} M_{\{1,2\}}$ , also  $\text{tp}(a/b'_0 M_{\{1,2\}})$  is isolated. Applying  $\sigma$ , also  $\text{tp}(\sigma(a)/b'_0 M_{\{2\}})$  is isolated, and, applying the TV Lemma and Lemma 3.18(i),

$$\begin{aligned} \text{tp}(\sigma(a)/b'_0 M_{\{2\}}) &\models \text{tp}(\sigma(a)/M_{\{0\}} M_{\{2\}}) \\ &\models \text{tp}(\sigma(a)/M_{\{0,1\}} M_{\{1,2\}}) \\ &\models \text{tp}(\sigma(a)/ab'_0 M_{\{1,2\}}), \end{aligned}$$

and so  $\text{tp}(a, \sigma(a)/M_{\{1,2\}}b'_0)$  is isolated, as required.  $\square$

### 3.6 Classification

**Lemma 3.30** (Constructible Models). *Let  $M \models T$ , let  $M_0 \prec M$  be a copy of the prime model, and let  $B$  be as in Subsection 3.1.*

*Let  $\widetilde{M}_0 \models \widehat{T}$  with  $\rho(\widetilde{M}_0) = M_0$ .*

*Then there exists  $\widetilde{M} \succ^* \widetilde{M}_0$  constructible over  $B\widetilde{M}_0$ , with  $\rho(\widetilde{M}) = M$ .*

*Proof.* Let  $I := \mathcal{P}^{\text{fin}}(B)$ .

Let  $(M_s)_{s \in I}$  be a constructible independent  $I$ -system as given by Lemma 3.23.

Let (by Lemma 3.6(a)(i))  $\widetilde{M}$  be an l-constructible model over  $M\widetilde{M}_0$ , and let  $\widetilde{M}_s = \rho^{-1}(M_s) \subseteq \widetilde{M}$ . By Lemma 3.6(b),  $\widetilde{M}_\emptyset = \widetilde{M}_0$  and  $\rho(\widetilde{M}) = M$ , and by Lemma 3.3,  $(\widetilde{M}_s)_s$  is an  $I$ - $\sim$ -system.

By Proposition 3.26,  $(\widetilde{M}_s)$  is a constructible independent system. By Lemma 3.11, each  $\widetilde{M}_{\{p\}}$  for  $p \in B$  is constructible over  $\widetilde{M}_\emptyset p$ , and so by Lemma 3.21,  $\widetilde{M} = \widetilde{M}_I$  is constructible over  $\widetilde{M}_\emptyset B = \widetilde{M}_0 B$ .  $\square$

**Theorem 3.31** (Classification). *A model  $\widetilde{M} \models \widehat{T}$  is determined up to isomorphism among models of  $\widehat{T}$  by*

- (i) *the isomorphism type of the lift  $\widetilde{M}_0 = \rho^{-1}(M_0)$  of a copy  $M_0 \prec M$  of the prime model, and*
- (ii) *the isomorphism type of  $M$  over  $M_0$ .*

*More explicitly: if  $\widetilde{M}^1, \widetilde{M}^2 \models \widehat{T}$ , if  $\widetilde{M}_0^1 \cong \widetilde{M}_0^2$  where  $\widetilde{M}_0^i$  is the lift  $\rho^{-1}(M_0^i)$  of a copy  $M_0^i \prec M^i$  of the prime model, and if the induced isomorphism  $M_0^1 \cong M_0^2$  extends to an isomorphism  $M^1 \cong M^2$ , then  $\widetilde{M}^1 \cong \widetilde{M}^2$ , in fact by an isomorphism extending the isomorphism  $\widetilde{M}_0^1 \cong \widetilde{M}_0^2$  (but not necessarily agreeing with the isomorphism  $M^1 \cong M^2$ ).*

$$\begin{array}{ccc} \widetilde{M}_0 & \hookrightarrow & \widetilde{M} \\ \downarrow & & \downarrow \\ M_0 & \hookrightarrow & M \end{array}$$

*Proof.* Given  $\widetilde{M} \models \widehat{T}$  and  $M_0 \prec M := \rho(\widetilde{M})$ , let  $B$  be as in Subsection 3.1.

Then  $\widetilde{M}$  is constructible and minimal over  $B\widetilde{M}_0$ , by Lemma 3.30 and the minimality of  $M$  over  $BM_0$ .

So let  $M^i$ ,  $\widetilde{M}_0^1 \cong \widetilde{M}_0^2$ , and  $M^1 \cong M^2$  be as in the statement. Let  $B^1$  be as in Subsection 3.1, and let  $B^2$  be the image in  $M^2$ . Then by the quantifier elimination,  $B^1\widetilde{M}_0^1 \equiv B^2\widetilde{M}_0^2$ , and by constructibility of  $\widetilde{M}^1$  over  $B^1\widetilde{M}_0^1$ , this extends to an elementary embedding  $\widetilde{M}^1 \prec \widetilde{M}^2$ ; but then by minimality of  $\widetilde{M}^2$  over  $B^2\widetilde{M}_0^2$ , the embedding is an isomorphism.  $\square$

*Remark 3.32.* We can also conclude that if  $M$  is strongly  $\aleph_1$ -homogeneous (e.g. if we take  $M$  to be saturated and uncountable), then  $\widetilde{M}$  is strongly  $\aleph_0$ -homogeneous over  $\widetilde{M}_0$ . Indeed, if  $\widetilde{a} \equiv_{\widetilde{M}_0} \widetilde{a}'$ , then by homogeneity we have  $B'$  such that  $B\widehat{a} \equiv_{M_0} B'\widehat{a}'$ , so  $B\widetilde{M}_0\widehat{a} \equiv B'\widetilde{M}_0\widehat{a}'$ ; but  $\widetilde{M}$  is also constructible and minimal over  $B\widetilde{M}_0\widehat{a}$  and over  $B'\widetilde{M}_0\widehat{a}'$ , so this extends to an automorphism.

Similarly, we obtain strong  $\aleph_0$ -homogeneity over an arbitrary countable  $*$ -submodel  $\widetilde{M}_1 \prec^* \widetilde{M}$ , replacing  $B$  with acl-bases over  $M_1$ .

Moreover, by Proposition 3.26, we similarly obtain strong  $\aleph_0$ -homogeneity over  $\widetilde{M}_{<|n|}$  for a  $\mathcal{P}(|n|)$ - $\sim$ -system in  $\widetilde{M}$ . Note that in the context of [BHH<sup>+</sup>14], and even in the specific example of pseudo-exponentiation, the corresponding results require a saturation hypothesis on  $M_s$ .

## 4 Exponential maps of semiabelian varieties

In this section, we apply our classification result Theorem 3.31, along with some arithmetic Kummer theory, to prove Theorem 1.1 and draw some related conclusions.

We actually work in slightly greater generality than Theorem 1.1, by allowing split semiabelian varieties. So throughout this section, we will suppose that  $\mathbb{G}(\mathbb{C})$  is the product  $A \times \mathbb{G}_m^n$  of a (possibly trivial) complex abelian variety and a (possibly trivial) algebraic torus.

Let  $\mathcal{O} := \text{End}(\mathbb{G})$  be the ring of algebraic endomorphisms of  $\mathbb{G}$ . Suppose  $\mathbb{G}$  and its endomorphisms are defined over  $k_0 \leq \mathbb{C}$ .

We first explain how we attach to the algebraic group  $\mathbb{G}$  a theory  $T$  satisfying the assumptions of the previous sections.

$\mathbb{G}$  can be viewed as a definable group in  $\text{ACF}_0$ , and as such inherits the structure of a finite Morley rank group. Explicitly, we consider  $\mathbb{G}(K)$ , for  $K$  an algebraically closed extension of  $k_0$ , as a structure in the language  $L$  consisting of a predicate for each  $k_0$ -Zariski-closed subset of each Cartesian power  $\mathbb{G}^n(K)$ . This structure is bi-interpretable with the field  $(K; +, \cdot, (c)_{c \in k_0})$  with parameters for  $k_0$ , and is a finite Morley rank group of rank  $\dim(\mathbb{G})$ . We let  $T$  be the theory of  $\mathbb{G}(\mathbb{C})$  in the language consisting of a predicate for each  $k_0$ -Zariski-closed subset of  $\mathbb{G}^n(\mathbb{C})$ . This is a commutative divisible group of finite Morley rank, admits quantifier elimination, and, by Lemma 4.1 below, every connected definable subgroup of  $\mathbb{G}^n$  is over  $k_0$ , so is defined over  $\emptyset$  in  $T$ .

### 4.1 $\mathcal{O}$ -module homomorphisms as models of $\widehat{T}$

By Proposition 2.18, the Lie exponential map  $\exp : L\mathbb{G} \rightarrow \mathbb{G}(\mathbb{C})$  has the structure of a model of  $\widehat{T}$ , which we denote by  $L\mathbb{G}$ . As a step towards proving Theorem 1.1, we prove in this subsection an abstract algebraic version of this.

Let  $\mathcal{O} := \text{End}(\mathbb{G})$  be the ring of algebraic (equivalently, definable) endomorphisms. The derivative at the identity  $L\eta$  of  $\eta \in \mathcal{O}$  is a  $\mathbb{C}$ -linear endomorphism of  $L\mathbb{G}$ , and we consider  $L\mathbb{G}$  as an  $\mathcal{O}$  module with this action.

**Lemma 4.1.** *(i) Any connected algebraic subgroup  $H \leq \mathbb{G}^n$  is the connected component of the kernel of an endomorphism  $\eta \in \text{End}(\mathbb{G}^n) \cong \text{Mat}_{n,n}(\mathcal{O})$ , and*

*(ii)  $LH \leq L\mathbb{G}^n$  is then the kernel of  $L\eta \in \text{End}_{\mathbb{C}}(L\mathbb{G}^n)$ .*

*Proof.* (i) By Poincaré's complete reducibility theorem, there exists an algebraic subgroup  $H'$  such that the summation map  $\Sigma : H \times H' \rightarrow \mathbb{G}^n$  is an isogeny. So say  $\theta : \mathbb{G}^n \rightarrow H \times H'$  is an isogeny such that  $\theta\Sigma = [m]$ , and let  $\pi_2 : H \times H' \rightarrow H'$  be the projection. Then  $\pi_2\theta\Sigma(h, h') = mh'$ , so  $\ker(\pi_2\theta)^o = \Sigma(H \times H'[m])^o = (H + H'[m])^o = H$ .

(ii)  $L\eta$  takes values in the discrete group  $\ker(\exp)^n$  on  $LH$ , so by connectedness and continuity  $L\eta$  is zero on  $LH$ . Conversely,  $\exp(\ker(L\eta))$  is a divisible subgroup of  $\ker(\eta)$ , and hence is contained in  $\ker(\eta)^o = H$ . So  $\ker(L\eta)$  is a subgroup of  $\exp^{-1}(H)$  containing  $LH$ ; but  $\ker(L\eta)$  is a  $\mathbb{C}$ -subspace so is connected, so  $\ker(L\eta) = LH$ . □



*Remark 4.2.* Lemma 4.1 (i) can fail for  $\mathbb{G}$  a semiabelian variety.

**Proposition 4.3.** *If  $K$  is an algebraically closed field extension of  $k_0$ , any surjective  $\mathcal{O}$ -module homomorphism  $\rho : V \rightarrow \mathbb{G}(K)$  from a divisible torsion-free  $\mathcal{O}$ -module  $V$  with finitely generated kernel is a model of  $\widehat{T}$ , where  $\widehat{H}$  is interpreted as the kernel of the action of  $\eta$  on  $V^n$  if  $H$  is the connected component of the kernel of  $\eta \in \text{End}(\mathbb{G}^n) \cong \text{Mat}_{n,n}(\mathcal{O})$ , and  $\rho_n(x) := \rho(x/n)$ .*

*Proof.* We appeal to Lemma 2.6. We will see in the course of the proof that  $\widehat{H}$  is indeed the divisible part of  $\rho^{-1}(H)$ , as assumed in that lemma, hence in particular that  $\widehat{H}$  is well-defined.

We use the following elementary principle, which we will call (\*): if  $A, B, F$  are subgroups of a torsion-free abelian group,  $A$  and  $B$  are divisible, and  $F$  is finitely generated, and if  $A \leq B + F$ , then  $A \leq B$ .

Suppose  $H = \ker(\eta)^o \leq \mathbb{G}^n$ , and  $\widehat{H} = \ker(\eta)$ . We show that  $\rho_k(\widehat{H}) = H$  for all  $k$ . By working in  $\mathbb{G}^n$ , we may assume  $n = 1$ . Let  $\Lambda := \ker \rho \leq V$ , and let  $\Lambda_0 \leq V$  be the divisible hull of  $\Lambda$ .

**Claim 4.4.**  $\eta(\Lambda_0) = \text{im } \eta \cap \Lambda_0$ .

*Proof.* First, note that  $\eta(\text{Tor}(\mathbb{G})) = \text{Tor}(\text{im } \eta)$ . Indeed, if  $n\eta(g) = 0$ , then  $ng \in \ker \eta$ , so  $mng \in (\ker \eta)^o$  for some  $m$ ; then by divisibility of  $(\ker \eta)^o$ , say  $h \in (\ker \eta)^o$  with  $mnh = mng$ . Then  $\eta(g - h) = \eta(g)$  and  $g - h \in \text{Tor}(\mathbb{G})$ .

Hence  $\text{im } \eta \cap \Lambda_0 \leq \eta(\Lambda_0) + \Lambda$ , so by (\*) already  $\text{im } \eta \cap \Lambda_0 \leq \eta(\Lambda_0)$ . The converse is immediate.  $\square$

Now since  $\Lambda$  is finitely generated,  $\eta(\Lambda)$  is a finite index subgroup of  $\text{im } \eta \cap \Lambda$ . By the snake lemma (see diagram), it follows that  $\rho(\widehat{H})$  is of finite index in  $\ker(\eta)$ . So by divisibility of  $\widehat{H}$ , we have  $\rho(\widehat{H}) = \ker(\eta)^o = H$ , and then  $\rho_k(\widehat{H}) = \rho(\widehat{H}) = H$  for all  $k$ . So (A6) holds.

$$\begin{array}{ccccccc}
& & & \Lambda & \longrightarrow & \Lambda \cap \text{im } \eta & \longrightarrow \cdots \\
& & & \downarrow & & \downarrow & \\
& & \widehat{H} & \longrightarrow & V & \longrightarrow & \text{im } \eta \longrightarrow 0 \\
& & \downarrow & & \downarrow \rho & & \downarrow \\
0 & \longrightarrow & \ker \eta & \longrightarrow & \mathbb{G} & \longrightarrow & \text{im } \eta \\
& & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \text{Finite} & \longrightarrow & 0 & & 
\end{array}$$

Hence  $\rho^{-1}(H) = \widehat{H} + \Lambda$ , so by (\*),  $\widehat{H}$  is the divisible part of  $\rho^{-1}(H)$ . Finally, we verify (A9). Let  $\text{pr} : G \rightarrow H$  be as in that axiom. By (A6),

$$\rho(\text{pr}(\widehat{G})) = \text{pr}(\rho(\widehat{G})) = \text{pr}(G) = H = \rho(\widehat{H}),$$

so  $\text{pr}(\widehat{G}) + \Lambda = \widehat{H} + \Lambda$ , so by (\*),  $\text{pr}(\widehat{G}) = \widehat{H}$ .  $\square$

## 4.2 Kummer theory

Suppose now that  $k_0$  is a number field.

Using this assumption, we may appeal to Kummer theory to reduce consideration of the prime model to consideration of the kernel. This is essentially the same argument as in [Gav08, Lemma 4].

Recall  $T = \text{Th}(\mathbb{G}(\mathbb{C}))$  in the language with a predicate for each subvariety defined over  $k_0$  of a cartesian power of  $\mathbb{G}$ .

**Lemma 4.5.** *Suppose  $\widetilde{M}_0 \models \widehat{T}$  with  $M_0 = \rho(\widetilde{M}_0) = \mathbb{G}(\overline{\mathbb{Q}})$ , the prime model of  $T$ . Then  $\widetilde{M}_0$  is constructible over  $\ker(\widetilde{M}_0)$ .*

*Proof.* Write  $\ker$  for  $\ker(\widetilde{M}_0)$ .

We use notation and results from Section A.

By Lemma 3.8, it suffices to show atomicity. Let  $\tilde{c} \in \widetilde{M}_0$ .

Let  $H + \zeta$  be the minimal torsion coset (see Section A.4) containing  $c$ . Then  $\tilde{c} \in \widehat{H} + \tilde{\zeta}$  for some  $\tilde{\zeta} \in \mathbb{Q}\ker$ . By translating, we may assume  $\tilde{\zeta} = 0$ , so then  $\tilde{c} \in \widehat{H}$  and  $H$  is the minimal torsion coset containing  $c$ .

By Proposition A.9, the image of the Kummer pairing is open,

$$Z_\infty := \langle \text{Gal}(k_0(c, \mathbb{G}[\infty])), c \rangle \leq_{\text{op}} T_\infty^H,$$

so  $nT_\infty^H \leq Z_\infty$  for some  $n$ , so

$$\text{tp}^T(c_n/\mathbb{G}[\infty]) \cup \bigcup_i \{c_i \in H\} \cup \bigcup_{k,m} \{[m]c_{km} = c_k\} \models \text{tp}^T(\tilde{c}/\mathbb{G}[\infty]).$$

So since  $\widehat{\ker} = G[\infty]$ , it follows by Lemma 2.16 that

$$\text{tp}^T(c_n/\mathbb{G}[\infty]) \cup \{\tilde{c} \in \text{grploc}(\tilde{c}/\ker)\} \models \text{tp}(\tilde{c}/\ker).$$

But  $\text{tp}^T(c_n/\mathbb{G}[\infty])$  is isolated since  $c_n \in \mathbb{G}(\overline{\mathbb{Q}})$ , so  $\text{tp}(\tilde{c}/\ker)$  is isolated as required.  $\square$

## 4.3 Categoricity and characterisation

We continue to assume that  $k_0$  is a number field.

Combining Lemma 4.5 with Theorem 3.31, and using uncountable categoricity of  $T$  to simplify the latter, we conclude:

**Theorem 4.6.** *A model  $\widetilde{M}$  of  $\widehat{T}$  is determined up to isomorphism over  $\ker(\widetilde{M})$  by*

- (i) *the isomorphism type of  $\ker(\widetilde{M})$ , equipped with all structure induced from  $\widehat{T}$*
- (ii) *the transcendence degree of  $K_M$ , where  $M \cong \mathbb{G}(K_M)$ .*

*Proof.* Suppose  $\widetilde{M}^1, \widetilde{M}^2 \models \widehat{T}$ , and  $\ker(\widetilde{M}^1) \cong \ker(\widetilde{M}^2)$  and  $\text{trd}(K_{M^1}) = \text{trd}(K_{M^2})$ . Let  $\widetilde{M}_0^i$  be the inverse image of  $M_0^i := \mathbb{G}(\overline{\mathbb{Q}}) \prec M^i$ . Then by Lemma 4.5 and the minimality of  $\mathbb{G}(\overline{\mathbb{Q}})$  over  $\emptyset$ , the isomorphism  $\ker(\widetilde{M}^1) \cong \ker(\widetilde{M}^2)$  extends to an isomorphism  $\widetilde{M}_0^1 \cong \widetilde{M}_0^2$ . The induced isomorphism  $M_0^1 \cong M_0^2$  extends to an isomorphism  $M^1 \cong M^2$ ; indeed, it induces a field automorphism of  $\overline{\mathbb{Q}}$  over

$k_0$ , which by the equality of transcendence degrees extends to an isomorphism  $K_{M^1} \cong K_{M^2}$ , inducing an isomorphism  $M^1 \cong M^2$ .

We conclude by Theorem 3.31.  $\square$

**Corollary 4.7.** *The model  $L\mathbb{G} \models \widehat{T}$  is the unique  $\widehat{L}$ -structure  $\widetilde{M}$  satisfying:*

(I)  $\widehat{T}$

(II)  $|\widetilde{M}| = 2^{\aleph_0}$

(III)  $\ker^{\widetilde{M}} \cong \ker^{L\mathbb{G}}$ , a partial  $\widehat{L}$ -isomorphism.

Moreover, for any such  $\widetilde{M}$ , the isomorphism of (III) extends to an isomorphism of  $\widetilde{M}$  with  $L\mathbb{G}$ .

**Theorem 4.8** (Theorem 1.1). *Suppose  $\rho, \rho' : L\mathbb{G} \rightarrow \mathbb{G}(\mathbb{C})$  are surjective  $\mathcal{O}$ -module homomorphisms,  $\ker \rho' = \ker \rho$ , and  $\rho' \upharpoonright_{\langle \ker \rho' \rangle_{\mathbb{Q}}} = \rho \upharpoonright_{\langle \ker \rho \rangle_{\mathbb{Q}}}$ .*

*Then there exists an  $\mathcal{O}$ -module automorphism  $\sigma \in \text{Aut}_{\mathcal{O}}(L\mathbb{G}/\ker \rho)$  and a field automorphism  $\tau \in \text{Aut}(\mathbb{C}/k_0)$  of  $\mathbb{C}$  fixing  $k_0$  such that  $\tau\rho' = \rho\sigma$ .*

*Proof.* Let  $\widetilde{M}$  and  $\widetilde{M}'$  be the corresponding  $\widehat{L}$  structures. By Proposition 4.3, they are models of  $\widehat{T}$ . By the QE and the assumption on the kernels, the structure induced on  $\ker \rho$  by the two structures is the same, and the transcendence degrees are both  $2^{\aleph_0}$ . So by Theorem 4.6,  $\widetilde{M}' \cong \widetilde{M}$  as  $\widehat{L}$ -structures, by an isomorphism fixing  $\ker \rho$ . Since the graphs of addition and of the action of each  $\eta \in \mathcal{O}$  are interpretations of appropriate  $\widehat{H}$ , this isomorphism induces an  $\mathcal{O}$ -module automorphism  $\sigma$  of  $L\mathbb{G}$ , and by the choice of language for  $T$  it induces a field automorphism  $\tau$  over  $k_0$ .  $\square$

Understanding the structure of  $\ker$  involves an understanding of the action of Galois on the torsion, which in general is known to be a hard problem. But let us highlight a strengthening of Theorem 4.6 in the case of the characteristic 0 multiplicative group:

**Theorem 4.9.** *Let  $\mathbb{G} = \mathbb{G}_m(\mathbb{C})$ . Then a model  $\widetilde{M}$  of  $\widehat{T}$  is determined up to isomorphism by the transcendence degree of the algebraically closed field  $K$  such that  $M \cong \mathbb{G}_m(K)$ , and the isomorphism type of  $\ker \rho$  as an abstract group.*

*Proof.* This is immediate from Theorem 4.6 once we see that the isomorphism type of  $\ker$  as a  $\widehat{L}$ -structure is determined by its isomorphism type as an abstract group. But this follows easily from the quantifier elimination and the fact from cyclotomic theory that any group automorphism of the roots of unity is a Galois automorphism.  $\square$

*Remark 4.10.* In the case of an elliptic curve  $\mathbb{G} = E$  there are only finitely many isomorphism types for a kernel with underlying group  $\langle \mathbb{Z}^2; + \rangle$  ([Gav08], [Bay10, Theorem 4.3.2]).

See also [Gav06, IV.6.3, IV.7.4] for some discussion of the higher dimensional situation.

*Question 4.11.* The assumption that  $k_0$  is a number field was used in Lemma 4.5. It is natural to ask whether this is essential. Does an appropriate version of Kummer theory go through for Abelian varieties over function fields? We are not aware of this question being fully addressed in the literature, but [Ber11, Theorem 5.4] goes some way toward it.

## 5 Further examples

In this section, we make some brief remarks on some other natural examples of Theorem 3.31.

### 5.1 Positive characteristic

We can not in general expect to improve on Theorem 3.31 in positive characteristic: if  $\mathbb{G}$  is the multiplicative group of a characteristic  $p > 0$  algebraically closed field, then the prime model is  $G(\mathbb{F}_p^{\text{alg}})$ , which is also the torsion group of  $\mathbb{G}$ . In this case, we recover the main theorem, 2.2, of [BZ11].

### 5.2 Manin kernels

In the theory DCF of differentially closed fields of characteristic 0, the Kolchin closures of the torsion of semiabelian varieties, also known as *Manin kernels*, are commutative divisible groups of finite Morley rank. A connected definable subgroup of such a Manin kernel is the Manin kernel of its Zariski closure, so Manin kernels of semiabelian varieties are rigid. Our classification theorem therefore applies to this case. By considering a local analytic trivialisation, a natural analytic model of  $\widehat{T}$  for  $\mathbb{G}$  a (non-isoconstant) Manin kernel can be given; this will be addressed in future work.

### 5.3 Meromorphic Groups

Let  $\mathbb{G}$  be a connected meromorphic group in the sense of [PS03], i.e. a connected definable group in the structure  $\mathcal{A}$  of compact complex spaces definable over  $\emptyset$  (equivalently, over  $\mathbb{C}$ ). By [PS03, Fact 2.10],  $\mathbb{G}$  can be uniquely identified with a complex Lie group.

Considering  $\mathbb{G}$  with its induced structure, it is a finite Morley rank group. Suppose  $\mathbb{G}$  is commutative and rigid. By the classification in [PS03] and the fact that any commutative complex linear algebraic group is a product of copies of  $\mathbb{G}_m$  and  $\mathbb{G}_a$ , there is a definable exact sequence of Lie groups

$$0 \rightarrow \mathbb{G}_m^n \rightarrow \mathbb{G} \rightarrow H \rightarrow 0$$

where  $H$  is a complex torus. It is also shown in [PS03] that  $\mathbb{G}$  is definable in a Kähler space; the latter may be considered in a countable language by [Moo05], so we may consider the language of  $\mathbb{G}$  to be the induced countable language. Let  $T = \text{Th}(\mathbb{G})$ .

In particular, in the case that  $\mathbb{G}$  is a complex semiabelian variety, we may take the language to be that induced from the field, as in Section 4 above.

Now let  $L\mathbb{G}$  be the analytic universal cover of the Lie group  $\mathbb{G}$ , considered as an  $\widehat{L}$ -structure as in Subsection 2.4.

By Proposition 2.18,  $L\mathbb{G} \models \widehat{T}$ . So by Theorem 3.31,  $L\mathbb{G}$  is the unique kernel-preserving extension of its restriction to the prime model  $\mathbb{G}_0$  of  $\mathbb{G}$ , which is a countable structure.

*Question 5.1.* Could the Kummer theory of Lemma 4.5 apply here? Concretely: is  $\rho^{-1}(\mathbb{G}_0)$  atomic over  $\ker$ ?

## A Kummer theory for $A \times \mathbb{G}_m^n$

In this appendix, we show that the results on Kummer theory for abelian varieties over number fields apply also to semiabelian varieties of the form  $A \times \mathbb{G}_m^n$  for  $A$  an abelian variety over a number field. This should perhaps be considered a known result, but we could find no complete proof in the literature.

Our approach owes much to Daniel Bertrand. In the case that  $\mathbb{G} = A$ , the Kummer theoretical result we require is precisely [Ber11, Theorem 5.2]; the purpose of this appendix is to show that this result holds also for  $A \times \mathbb{G}_m^n$ , with a mostly parallel proof. As in that article, the method we apply is essentially that of Ribet's paper [Rib79].

We should note that for general semiabelian varieties over number fields, Kummer theory is known to fail due to the existence of *deficient points* - see [JR87].

### A.1 Finiteness theorems for abelian varieties

Let  $A$  be an abelian variety over a number field  $k_0$ , let  $T_l^A := \varprojlim_n A[l^n]$  for  $l$  prime be the Tate modules, and let  $T_\infty^A := \varprojlim_n A[n] = \prod_l T_l^A$ .

The following result on Galois cohomology is a consequence of Serre's uniform version of Bogomolov's result on homotheties. Here and below,  $H^i$  refers to continuous group cohomology.

**Fact A.1.**  $H^1(\text{Gal}(k_0(A[\infty])/k_0), A[n])$  has uniformly bounded finite exponent, i.e. there exists  $c > 0$  such that for all  $n > 0$ , we have  $c \cdot H^1(\text{Gal}(k_0(A[\infty])/k_0), A[n]) = 0$ .

*Proof.* Let  $G_\infty := \text{Gal}(k_0(A[\infty])/k_0)$ .

Note that  $H^1(G_\infty, A[n])$  admits a prime power decomposition as  $\prod_i H^1(G_\infty, A[l_i^{k_i}])$  where  $n = \prod_i l_i^{k_i}$ .

By [Ser00, Théorème 2', "Résumé des cours de 1985-1986", proved in "Lettre à Ken Ribet du 7/3/1986" in the same volume], there exists  $M > 0$  such that every  $M$ th power homothety is in the image of  $G_\infty$ , i.e. any element of  $\hat{Z}^* = \prod_l \mathbb{Z}_l^*$  which is an  $M$ th power in that group is the action on  $T_\infty^A$  of some element of  $G_\infty$ .

In particular, there is  $\sigma \in G_\infty$  which acts on  $T_l^A$  as multiplication by  $2^M$  for  $l \neq 2$ , and acts on  $T_2^A$  as the identity. Then  $\sigma$  is central in  $G_\infty$ , so by Sah's Lemma,  $H^1(G_\infty, T_\infty^A)$  and each  $H^1(G_\infty, A[n])$  are annihilated by  $\sigma - 1$ . Then if  $l$  is an odd prime which does not divide  $2^M - 1$ , so  $2^M - 1 \in \mathbb{Z}_l^*$ , we have  $H^1(G_\infty, A[l^k]) = 0$  for all  $k$ .

Let  $2 = l_0, l_1, \dots, l_s$  be the remaining primes, and let  $p \notin \{l_0, \dots, l_s\}$  be another prime. Then by the same argument,  $p^M - 1$  annihilates each  $H^1(G_\infty, A[l_i^k])$ .

So  $p^M - 1$  annihilates each  $H^1(G_\infty, A[n])$ . □

The second ingredient is the following result of Faltings, sometimes referred to, after Lang, as Finiteness I [Lan91, IV.2]. Here, a  $k_0$ -isogeny is an isogeny defined over  $k_0$ ; similarly for  $k_0$ -isomorphism.

**Fact A.2** (Faltings). *The algebraic groups which are  $k_0$ -isogenous to  $A$  fall into finitely many  $k_0$ -isomorphism classes.*

## A.2 Generalisations to $A \times \mathbb{G}_m^n$

Let  $\mathbb{G} = A \times \mathbb{G}_m^n$  with  $A$  an abelian variety over a number field  $k_0$ . We check that the results of the previous section imply the corresponding results for  $\mathbb{G}$ .

**Lemma A.3.**  *$H^1(\text{Gal}(k_0(\mathbb{G}[\infty])/k_0), A[n])$  has uniformly bounded finite exponent, i.e. there exists  $c > 0$  such that for all  $n > 0$ , we have  $c \cdot H^1(\text{Gal}(k_0(\mathbb{G}[\infty])/k_0), A[n]) = 0$ .*

*Proof.* By Hilbert 90,  $H^1(\text{Gal}(k_0(\mathbb{G}[\infty])/k_0), \mu_m) = 0$ . Meanwhile,  $k_0(\mathbb{G}[\infty]) = k_0(A[\infty])$  since the multiplicative roots of unity are rational over  $k_0(A[\infty])$ , via a Weil pairing.

So

$$\begin{aligned} H^1(\text{Gal}(k_0(\mathbb{G}[\infty])/k_0), \mathbb{G}[n]) &\cong H^1(\text{Gal}(k_0(\mathbb{G}[\infty])/k_0), A[n]) \\ &= H^1(\text{Gal}(k_0(A[\infty])/k_0), A[n]), \end{aligned}$$

and we conclude by Fact A.1.  $\square$

**Lemma A.4.** *The algebraic groups which are  $k_0$ -isogenous to  $\mathbb{G}$  fall into finitely many  $k_0$ -isomorphism classes.*

*Proof.* Let  $T := \mathbb{G}_m^n$ .

Recall (see e.g. [Ser58, 10]) that a semiabelian variety which falls into an exact sequence  $0 \rightarrow T \rightarrow S \rightarrow A \rightarrow 0$  corresponds to a point in the  $n$ th power of the dual abelian variety of  $A$ ,

$$\text{Ext}(A, T) \cong \text{Ext}(A, \mathbb{G}_m)^n \cong (A^\vee)^n.$$

Let  $\mathbb{G}'$  be  $k_0$ -isogenous to  $\mathbb{G}$ , so  $\mathbb{G}' \cong \mathbb{G}/Z$  for  $Z \leq \mathbb{G}$  a finite subgroup defined over  $k_0$ . Since  $\mathbb{G}/(Z \cap T)$  is  $k_0$ -isomorphic to  $\mathbb{G}$ , we may assume  $Z \cap T = 0$ .

Let  $\pi_1 : \mathbb{G} \rightarrow A$  and  $\pi_2 : \mathbb{G} \rightarrow T$  be the projections of the product. Let  $A' := A/\pi_1(Z)$  be the quotient abelian variety. So  $\mathbb{G}'$  is an extension of  $A'$  by  $T$ , and so  $\mathbb{G}'$  corresponds to an element  $[\mathbb{G}']$  of  $\text{Ext}(A', T) \cong (A'^\vee)^n$ .

**Claim A.5.**  *$[\mathbb{G}']$  is a torsion element of  $\text{Ext}(A', T)$ .*

*Proof.* Let  $k$  be the exponent of the finite group  $\pi_2(Z) \leq T$ . Then the  $k$ -fold Baer sum  $[k]\mathbb{G}'$  of  $\mathbb{G}'$  in  $\text{Ext}(A', T)$  is split. Indeed,  $[k]\mathbb{G}'$  is the  $k$ -fold fibre product of  $\mathbb{G}'$  over  $A'$ , quotiented by the subgroup  $\Sigma := \{\sum_i \alpha_i = 0 \mid \alpha_i \in T\} \leq T^k \leq A'^k$ . Then the trivialisation  $x \mapsto (x, 0)$  of  $\mathbb{G} = A \times T$  induces a trivialisation of  $[k]\mathbb{G}'$ ,  $x + \pi_1(Z) \mapsto ((x, 0) + Z, \dots, (x, 0) + Z) + \Sigma$ ; this is well-defined as  $((x, 0) + Z) - ((x + \pi_1\zeta, 0) + Z) = (0, \pi_2\zeta) + Z$ , and  $(\pi_2\zeta, \dots, \pi_2\zeta) \in \Sigma$  since  $k\pi_2\zeta = 0$ .  $\square$

Now since  $\mathbb{G}'$  is defined over  $k_0$ , so is  $A'$  and so is the torsion point  $[\mathbb{G}']$  of  $(A'^\vee)^n$ . By Fact A.2, there are only finitely many such  $A'$  up to  $k_0$ -isomorphism, and by Mordell-Weil each has only finitely many  $k_0$ -rational torsion points. Hence, there are only finitely many possibilities for  $\mathbb{G}'$  up to  $k_0$ -isomorphism.  $\square$

### A.3 Group structure of $\mathbb{G}(k_0(\mathbb{G}[\infty]))$

**Definition A.6.** If  $\Gamma'$  is a subgroup of an abelian group  $\Gamma$ , let  $\text{pureHull}_\Gamma(\Gamma') := \{\gamma \in \Gamma \mid \exists n > 0. n\gamma \in \Gamma'\} \leq \Gamma$ .

An abelian group  $\Gamma$  is *locally free modulo torsion* if for any finitely generated subgroup  $\Gamma' \leq \Gamma$ , there exists  $m$  such that  $m \cdot \text{pureHull}_\Gamma(\Gamma') \leq \Gamma' + \text{Tor}(\Gamma)$ .

Now let  $k_0$  be a number field, let  $A$  be an abelian variety over  $k_0$ , and let  $\mathbb{G} = A \times \mathbb{G}_m^n$  be the product with an algebraic torus. Let  $k_\infty := k_0(\mathbb{G}[\infty])$ .

**Lemma A.7.**  $\mathbb{G}(k_\infty)$  is locally free modulo torsion.

*Remark A.8.* By countability of  $\mathbb{G}(k_\infty)$  and a theorem of Pontryagin [Fuc70, 19.1], an equivalent statement is that the quotient group  $\mathbb{G}(k_\infty)/\mathbb{G}[\infty]$  is free abelian. For  $\mathbb{G}$  an abelian variety over a number field, this is proven by Larsen in [Lar05]. This lemma generalises that result, using similar techniques.

*Proof.* Let  $\Gamma \leq \mathbb{G}(k_\infty)$  be a finitely generated subgroup. Replacing  $k_0$  by the number field  $k_0(\Gamma)$  if necessary, we assume  $\Gamma \leq \mathbb{G}(k_0)$ .

First, we see that  $\mathbb{G}(k_0) = A(k_0) \times \mathbb{G}_m^n(k_0)$  is free modulo torsion. We use Dirichlet's Unit theorem to examine the group structure of  $\mathbb{G}_m(k_0) = k_0^*$ . Here, we are following [Zil06, Lemma 2.1].

Let  $\mathcal{O}_{k_0}$  be the ring of integers of  $k_0$ . By Dirichlet's Unit theorem,  $\mathcal{O}_{k_0}^*$  is finitely generated. Recall that  $\mathcal{O}_{k_0}$  is a Dedekind domain and the fractional ideals,  $\text{Id}(\mathcal{O}_{k_0})$ , form a free abelian group with generators the prime ideals. We have an exact sequence

$$1 \longrightarrow \mathcal{O}_{k_0}^* \longrightarrow k_0^* \xrightarrow{\theta} \text{Id}(\mathcal{O}_{k_0}),$$

where  $\theta(x) := x\mathcal{O}_{k_0}$ . The image of  $\theta$  is a subgroup of a free abelian group, so is free abelian.

Meanwhile,  $A(k_0)$  is finitely generated by the Mordell-Weil theorem. So  $\mathbb{G}(k_0)$  is an extension of a free abelian group by a finitely generated group, so the quotient by the torsion is an extension of free abelian by free abelian, so is free abelian. Hence  $\mathbb{G}(k_0)$  is locally free modulo torsion.

So say  $m$  is such that  $m \cdot \text{pureHull}_{\mathbb{G}(k_0)}(\Gamma) \leq \Gamma + \mathbb{G}[\infty]$ .

Meanwhile, by Lemma A.3, say  $c \cdot H^1(\text{Gal}(k_\infty/k_0), \mathbb{G}[n]) = 0$  for all  $n$ .

We conclude by showing  $mc \cdot \text{pureHull}_{\mathbb{G}(k_\infty)}(\Gamma) \leq \Gamma + \mathbb{G}[\infty]$ .

Indeed, suppose  $\gamma \in \text{pureHull}_{\mathbb{G}(k_\infty)}(\Gamma)$ , say  $\gamma \in \mathbb{G}(k_\infty)$  and  $n\gamma \in \Gamma \leq \mathbb{G}(k_0)$ . Then  $\theta(\sigma) := \sigma\gamma - \gamma$  yields an element of  $H^1(\text{Gal}(k_\infty/k_0), \mathbb{G}[n])$ . So  $c\theta$  is a coboundary, so there is  $\zeta \in \mathbb{G}[n]$  such that  $c(\sigma\gamma - \gamma) = \sigma\zeta - \zeta$  for all  $\sigma \in \text{Gal}(k_\infty/k_0)$ , so  $c\gamma - \zeta \in \mathbb{G}(k_0)$ .

So  $c\gamma - \zeta \in \text{pureHull}_{\mathbb{G}(k_0)}(\Gamma)$ , so  $mc\gamma \in \Gamma + \mathbb{G}[\infty]$ .  $\square$

### A.4 Openness

Let  $\mathbb{G} = A \times \mathbb{G}_m^n$  as above. Let  $\mathcal{O} := \text{End}(\mathbb{G}) \cong \text{End}(A) \times \text{End}(\mathbb{G}_m^n)$ . By taking a finite field extension if necessary, we assume that each  $\eta \in \mathcal{O}$  is defined over the number field  $k_0$ .

We define the Kummer pairings for  $\mathbb{G}$  as follows: if  $k \geq k_0$ , and  $\gamma \in \mathbb{G}(k)$  and  $\sigma \in \text{Gal}(k)$ , let  $\langle \sigma, \gamma \rangle_n := \sigma\alpha - \alpha \in \mathbb{G}[n]$  for any  $\alpha \in \mathbb{G}(k)$  with  $n\alpha = \gamma$ , and let  $\langle \sigma, \gamma \rangle := (\langle \sigma, \gamma \rangle_n)_n \in T_\infty^\mathbb{G}$ .

A *torsion coset* in  $\mathbb{G}$  is the translate  $H + \zeta$  of a connected algebraic subgroup  $H \leq \mathbb{G}$  by a torsion point  $\zeta \in \mathbb{G}[\infty]$ .

By considering the torsion group, one sees that  $T_\infty^H$  for such an  $H$  is isomorphic to a finite power of  $\hat{\mathbb{Z}}$ , and so a subgroup  $Z$  of  $T_\infty^H$  is open in the profinite topology,  $Z \leq_{\text{op}} T_\infty^H$ , if and only if it is of finite index.

**Proposition A.9.** *Let  $\gamma \in \mathbb{G}(k_\infty)$ . Suppose  $H + \zeta$  is the minimal torsion coset containing  $\gamma$ . Then  $Z_\infty := \langle \text{Gal}(k_\infty), \gamma \rangle \leq_{\text{op}} T_\infty^H \leq T_\infty^{\mathbb{G}}$ .*

*Remark A.10.* In the case that  $\mathbb{G}$  is an abelian variety, this is exactly [Ber11, Theorem 5.2].

*Proof.* Since  $\langle \text{Gal}(k_\infty), \zeta \rangle = 0$ , by shifting by  $\zeta$  we may assume  $\gamma \in H$ .

Replacing  $k_0$  with  $k_0(\gamma)$  if necessary, we may assume  $\gamma \in \mathbb{G}(k_0)$ .

By Lemma 4.1 and the assumption that the endomorphisms are over  $k_0$ , we have that  $H$  is defined over  $k_0$ . So since  $H$  is divisible,  $Z_\infty \leq T_\infty^H$ . It remains to see that the index is finite.

Now  $\mathbb{G}(k_\infty)$  is an  $\mathcal{O}$ -submodule of  $\mathbb{G}(\bar{\mathbb{Q}})$  by the assumption that the endomorphisms are over  $k_0 \leq k_\infty$ , and  $\mathcal{O}\gamma$  is a finitely generated subgroup since  $\mathcal{O}$  is finitely generated. So by Lemma A.7, say  $m > 0$  is such that  $m \cdot \text{pureHull}_{\mathbb{G}(k_\infty)}(\mathcal{O}\gamma) \leq \mathcal{O}\gamma + \mathbb{G}[\infty]$ .

For  $n > 0$ , let  $Z_n := \langle \text{Gal}(k_\infty), \gamma \rangle_n \leq \mathbb{G}[n]$ . Note that  $Z_n$  is defined over  $k_0$ ; indeed, if  $\sigma \in \text{Gal}(k_\infty)$  and  $\tau \in \text{Gal}(k_0)$ , then  $\sigma\tau^{-1} \in \text{Gal}(k_\infty)$  and

$$\left\langle \sigma\tau^{-1}, \gamma \right\rangle_n = \tau\sigma\tau^{-1}\alpha - \alpha = \tau(\sigma(\tau^{-1}\alpha) - \tau^{-1}\alpha) = \tau \langle \sigma, \gamma \rangle_n$$

(where  $n\alpha = \gamma$ , and hence  $n\tau^{-1}\alpha = \gamma$ ). So  $(Z_n)^\tau = Z_n$ .

So by Lemma A.4, the isogenous groups  $B_n := \mathbb{G}/Z_n$  fall into finitely many  $k_0$ -isomorphism classes. Therefore we may find  $N$  such that for any  $n$ , there exists a  $k_0$ -isogeny  $\theta_n : B_n \rightarrow \mathbb{G}$  of degree  $\deg \theta_n := |\ker \theta_n|$  dividing  $N$ .

We conclude the proof of the Proposition by showing that for any  $n$ , the index  $[H[n] : Z_n]$  divides  $N \cdot |\mathbb{G}[m]|$ .

Indeed, let  $\eta \in \mathcal{O}$  be the composition  $\eta(x) := \theta_n(x/Z_n)$  of  $\theta_n$  with the quotient map. Suppose  $n\beta = \gamma$ . Then  $n\eta\beta = \eta\gamma$ . But  $\eta\beta$  is  $\text{Gal}(k_\infty)$ -invariant; indeed,  $Z_n \leq \ker(\eta)$  and  $\eta$  is defined over  $k_0 \leq k_\infty$ , so

$$\sigma\eta\beta = \eta\sigma\beta = \eta(\beta + \langle \sigma, \gamma \rangle_n) = \eta\beta.$$

So  $\eta\beta \in \text{pureHull}_{\mathbb{G}(k_\infty)}(\mathcal{O}\gamma)$ , so  $m\eta\beta \in \mathcal{O}\gamma + \mathbb{G}[\infty]$ . So  $m\eta\gamma \in n\mathcal{O}\gamma + \mathbb{G}[\infty]$ , so  $k(m\eta - n\eta')\gamma = 0$  for some  $k > 0$  and some  $\eta' \in \mathcal{O}$ . So by the choice of  $H$ , we have  $m\eta = n\eta'$  on  $H$ .

Hence  $m\eta(H[n]) = 0$ , i.e.  $\theta_n(H[n]/Z_n) \leq \mathbb{G}[m]$ , and hence

$$\begin{array}{l|l} [H[n] : Z_n] & |\ker \theta_n| \cdot |\mathbb{G}[m]| \\ & | N \cdot |\mathbb{G}[m]|. \end{array}$$

□



## References

- [Bay10] Martin Bays. *Categoricity Results for Exponential Maps of 1-Dimensional Algebraic Groups & Schanuel Conjectures for Powers and the CIT*. PhD thesis, Oxford University, 2010. URL: <http://wwwmath.uni-muenster.de/u/baysm/dist/thesis/>.
- [BBP16] Franck Benoist, Elisabeth Bouscaren, and Anand Pillay. Semiabelian varieties over separably closed fields, maximal divisible subgroups, and exact sequences. *J. Inst. Math. Jussieu*, 15(1):29–69, 2016.
- [Ber11] Daniel Bertrand. Galois descent in Galois theories. In *Arithmetic and Galois theories of differential equations*, volume 23 of *Sémin. Congr.*, pages 1–24. Soc. Math. France, Paris, 2011.
- [BGH14] Martin Bays, Misha Gavrilovich, and Martin Hils. Some definability results in abstract Kummer theory. *Int. Math. Res. Not. IMRN*, (14):3975–4000, 2014.
- [BHH<sup>+</sup>14] Martin Bays, Bradd Hart, Tapani Hyttinen, Meeri Kesälä, and Jonathan Kirby. Quasiminimal structures and excellence. *Bull. Lond. Math. Soc.*, 46(1):155–163, 2014.
- [Bue96] Steven Buechler. *Essential stability theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1996.
- [BZ11] Martin Bays and Boris Zilber. Covers of multiplicative groups of algebraically closed fields of arbitrary characteristic. *Bull. Lond. Math. Soc.*, 43(4):689–702, 2011.
- [Fuc70] László Fuchs. *Infinite abelian groups. Vol. I*. Pure and Applied Mathematics, Vol. 36. Academic Press, New York, 1970.
- [Gav06] Misha Gavrilovich. *Model theory of the universal covering spaces of complex algebraic varieties*. PhD thesis, Oxford University, 2006.
- [Gav08] Misha Gavrilovich. A remark on transitivity of Galois action on the set of uniquely divisible abelian extensions in  $\text{Ext}^1(E(\overline{\mathbb{Q}}), \Lambda)$ . *K-Theory*, 38(2):135–152, 2008.
- [Har87] Bradd Hart. An exposition of OTOP. In *Classification theory (Chicago, IL, 1985)*, volume 1292 of *Lecture Notes in Math.*, pages 107–126. Springer, Berlin, 1987.
- [JR87] Olivier Jacquinot and Kenneth A. Ribet. Deficient points on extensions of abelian varieties by  $\mathbf{G}_m$ . *J. Number Theory*, 25(2):133–151, 1987.
- [Lac73] Alistair H. Lachlan. On the number of countable models of a countable superstable theory. In *Logic, methodology, philosophy of science 4*, pages 45–56. 1973.
- [Lan91] Serge Lang. *Number theory. III*, volume 60 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1991. Diophantine geometry.

- [Lar05] Michael Larsen. A mordell-weil theorem for abelian varieties over fields generated by torsion points. 2005. arXiv:math/0503378v1 [math.NT].
- [Las85] Daniel Lascar. Les groupes  $\omega$ -stables de rang fini. *Trans. Amer. Math. Soc.*, 292(2):451–462, 1985.
- [Moo05] Rahim N. Moosa. The model theory of compact complex spaces. In *Logic Colloquium '01*, volume 20 of *Lect. Notes Log.*, pages 317–349. Assoc. Symbol. Logic, Urbana, IL, 2005.
- [PS03] Anand Pillay and Thomas Scanlon. Meromorphic groups. *Trans. Amer. Math. Soc.*, 355(10):3843–3859, 2003.
- [Rib79] Kenneth A. Ribet. Kummer theory on extensions of abelian varieties by tori. *Duke Math. J.*, 46(4):745–761, 1979.
- [Ser58] Jean-Pierre Serre. Quelques propriétés des variétés abéliennes en caractéristique  $p$ . *Amer. J. Math.*, 80:715–739, 1958.
- [Ser00] Jean-Pierre Serre. *Œuvres. Collected papers. IV*. Springer-Verlag, Berlin, 2000. 1985–1998.
- [She90] S. Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1990.
- [Zil02] B. Zilber. Model theory, geometry and arithmetic of the universal cover of a semi-abelian variety. In *Model theory and applications*, volume 11 of *Quad. Mat.*, pages 427–458. Aracne, Rome, 2002.
- [Zil06] Boris Zilber. Covers of the multiplicative group of an algebraically closed field of characteristic zero. *J. London Math. Soc. (2)*, 74(1):41–58, 2006.