Pseudofinite-dimensional Schrödinger representations

M. Bays & B. Hart

June 30, 2016

Abstract

Following Zilber [Zil16], we obtain the Schrödinger representation of the (3-dimensional) Heisenberg algebra on the tempered distributions as an ultralimit of finite-dimensional representations of certain finite subgroups of the Heisenberg group. Moreover, we see that the corresponding Weil representation can be obtained this way.

0.1 Schwartz space and tempered distributions

Let $\mathcal{S}$ be the Schwartz space of complex-valued functions on $\mathbb{R}$ with rapidly decreasing absolute value.

Recall that the topology on $\mathcal{S}$ is that given by the seminorms

$$p_{n,m}(\phi) = \sup_x \mid x^n \delta^m \phi(x) \mid .$$

Recall also that the subspace $\mathcal{D} \subseteq \mathcal{S}$ of compactly supported smooth functions is dense in $\mathcal{S}$.

Let $\mathcal{S}'$ be the space of tempered distributions, the continuous dual space of $\mathcal{S}$. Write the corresponding sesquilinear pairing as $\langle T, \phi \rangle := T(\overline{\phi})$ for $T \in \mathcal{S}$ and $\phi \in \mathcal{S}'$. We have the seminorms

$$\|T\|_\phi := \mid \langle T, \phi \rangle \mid$$

for $\phi \in \mathcal{S}$. A subset B of $\mathcal{S}'$ is bounded iff each $\|\cdot\|_\phi$ is bounded on B. We consider bounded subsets with the topology induced from the $\|\cdot\|_\phi$. In fact this topology coincides with that induced from the strong topology on $\mathcal{S}'$, but we will not explicitly use or define the strong topology. Closed bounded subsets of $\mathcal{S}'$ are compact [Sch66, Chap.III,Thm.XII].

Recall [Rudin 7.12(d)] that every measurable function $f : \mathbb{R} \to \mathbb{C}$ of polynomial growth, i.e. bounded in absolute value by some polynomial, induces a tempered distribution

$$\langle f, \phi \rangle := \int_{\mathbb{R}} f(x)\overline{\phi}(x)dx.$$

In particular this induces embeddings of $\mathcal{S}$ and $\mathcal{D}$ into $\mathcal{S}'$, which are dense by [Sch66, Chap.III Thm.XV].

The space of distributions $\mathcal{D}'$ is the continuous dual of $\mathcal{D}$. We have topological embeddings

$$\mathcal{D} \subseteq \mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{D}'.$$

Recall that a bilinear map $u : X \times Y \to Z$ of topological vector spaces is *hypocontinuous* if each $u(x, \cdot)$ and $u(\cdot, y)$ are continuous, and $\{u(x, \cdot) \mid x \in B\}$ is equicontinuous for any bounded $B \subseteq X$, so in particular the restriction of the pairing to $B \times Y$ is continuous, and similarly for $B' \subseteq Y$ bounded.

By [Sch66, Chap.III,Thm.XI;Chap.VII,Sect.4], the pairings $\mathcal{D} \times \mathcal{D}' \to \mathbb{C}$ and $\mathcal{S} \times \mathcal{S}' \to \mathbb{C}$ are hypocontinuous.

Let $\mathcal{E}$ be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{C}$ with the topology of compact uniform convergence of all derivatives. By [Sch66, Chap.V,Thm.III], multiplication of functions and distributions is hypocontinuous as a function $\mathcal{E} \times \mathcal{D}' \to \mathcal{D}'$.

0.2 Metric logic structure on $\mathcal{S}'$

We view $\mathcal{S}'$ as the union of its closed bounded subsets (forming a "bornology"). $\mathcal{S}'$ is not covered by any countable set of bounded sets, nor is $\mathcal{S}'$ the union of a chain of bounded sets.

Each closed bounded $B \subseteq \mathcal{S}'$ is compact, and moreover is a complete metric space. In fact, a single metric suffices to metrise any B. Indeed, $\mathcal{S}$ is separable, so let $(\psi_i^0)_{i<\omega}$ be a countable topological basis. Concretely, we can take $\psi_i^0(x) := x^i e^{-x^2}$. Then define on $\mathcal{S}'$

$$d(T, T') := \sum_i 2^{-i} \min(1, |\left\langle T - T', \psi_i^0 \right\rangle|).$$

Lemma 0.1. *Let $B \subseteq \mathcal{S}'$ be closed and bounded. Then d is a complete metric, metrising the topology of B.*

Proof. The topology on B is induced by the pseudonorms $\| \cdot \|_\phi$ and is complete with respect to them, so it suffices to see that for any $\phi \in \mathcal{S}$ and any ϵ, there exists δ such that for $T, T' \in B$,

$$d(T, T') < \delta \Rightarrow |\left\langle T - T', \phi \right\rangle| < \epsilon.$$

But indeed, the pairing is hypocontinuous, so $\langle T, . \rangle$ is equicontinuous on $B - B$; hence for a good enough approximation to ϕ as a finite linear combination ϕ_0 of the ψ_i^0, $|\langle T - T', \phi - \phi_0 \rangle| < \epsilon/2$ for all $T, T' \in B$. Then take δ such that

$$d(T, T') < \delta \Rightarrow |\left\langle T - T', \phi_0 \right\rangle| < \epsilon/2.$$

□

Compact metric spaces are the analogue in continuous logic of finite structures in discrete logic, and most notions from continuous logic degenerate in such spaces. In particular, the ultraproduct of subspaces may be identified with the topological ultralimit, as we now explain.

Consider a topological space X which is a union of compact metric subspaces $B \in \mathcal{B}$, and suppose $\mathcal{B}$ is closed under finite unions. Call $Y \subseteq X$ bounded if $Y \subseteq B$ for some $B \in \mathcal{B}$. The example we have in mind is $X = \mathcal{S}'$, with $\mathcal{B}$ the set of its closed bounded subsets. Now forget the topology on X, but retain the topology on each B. For $Y_i \subseteq B \in \mathcal{B}$ and an ultrafilter $\mathcal{U}$, recall $\lim_{i \to \mathcal{U}} Y_i = \{\lim_{i \to \mathcal{U}} y_i \mid y_i \in Y_i\} \subseteq B$. Now for $Y_i \subseteq X$, define $\lim_{i \to \mathcal{U}} Y_i = \bigcup_{B \in \mathcal{B}} \lim_{i \to \mathcal{U}} (Y_i \cap B)$.

For $X = \mathcal{S}'$, we will use this notion of ultralimit even though it does not agree with the usual topological ultralimit for even the strong topology

on X, since unbounded ultrafilters in X can converge; for our purposes such convergence is pathological, so we restrict to boundeds.

Now suppose $Y_i \subseteq X$ are subspaces such that $\lim_{i\to\mathcal{U}} Y_i = X$. If the Y_i are equipped with interpretations of a relation $R^{Y_i} \subseteq Y_i^n$, define the induced relation on X by $R^X := \lim_{i\to\mathcal{U}} R^{Y_i} \subseteq X^n$. Then this notion of ultralimit of structures agrees with the notion of ultraproduct of metric structures, in the following sense. Consider (Y_i, R^{Y_i}) as a metric structure in the language with a sort for each $B \in \mathcal{B}$, interpreted in Y_i as $Y_i \cap B$, and a real-valued metric predicate R'_B on each sort, interpreted in Y_i as the distance predicate $R'_B(x) = d(x, R^{Y_i} \cap B^n) = \inf_{y\in Y_i\cap B^n} d(x, y)$ (where we define d on X^n by $d(x, y) = max_i d(x_i, y_i)$, say). Then the metric ultraproduct is isomorphic to (X, R^X) considered as a multisorted metric structure in the same way.

We apply this to (partial) operators $X \to X$ by considering their graphs, so we define $F^X(\lim_{i\to\mathcal{U}} y_i) := \lim_{i\to\mathcal{U}} F^{Y_i}(y_i)$ whenever $\{y_i\}_i$ is bounded, $F^{Y_i}(y_i)$ are defined, and $\{F^{Y_i}(y_i)\}_i$ is bounded, if this definition yields a well-defined partial operator.

Say f^{Y_i} are provably bounded if $\{F^{Y_i}(y_i)\}_i$ is bounded whenever $\{y_i\}_i$ is bounded and the $F^{Y_i}(y_i)$ are defined. Note that the composition of provably bounded operators is provably bounded.

Metric logic admits a Łos theorem, but the only consequence of this we will need is the following weak statement for exact equality of compositions of operators, which follows directly from the above definitions.

Lemma 0.2. *Suppose $F_j^{Y_i}$ are partial operators, and $F_1^{Y_i} \circ F_2^{Y_i} = F_3^{Y_i}$ holds $\mathcal{U}$-often, and suppose that the ultralimits F_1^X and F_2^X are well-defined partial operators. Suppose also that the $F_2^{Y_i}$ are provably bounded. Then F_3^X is a well-defined partial operator, and $F_1^X \circ F_2^X = F_3^X$.*

Proof. Restricting the index set, we may assume the equality holds for all i. Let $\{y_i\}_i$ be bounded with $y_i \in \mathrm{dom}(F_3^{Y_i})$. Then $\{F_2^{Y_i}(y_i)\}_i$ is bounded by provable boundedness, so $\lim_{i\to\mathcal{U}} F_1^{Y_i}(F_2^{Y_i}(y_i)) = F_1^X(\lim_{i\to\mathcal{U}} F_2^{Y_i}(y_i)) = F_1^X(F_2^X(\lim_{i\to\mathcal{U}} y_i))$. □

0.3 Schrödinger representations

The 3-dimensional real Heisenberg Lie algebra η is the central extension of the trivial Lie algebra on $\mathbb{R}^2$ via the canonical symplectic structure on $\mathbb{R}^2$; it can be defined by saying it has basis (P, Q, E), with E being central and $[Q, P] = E$. The infinitesimal Schrödinger representation (with respect to Q) of η on the Schwartz space $\mathcal{S} \subseteq L^2(\mathbb{R})$ is given by $P\phi(x) = -\phi'(x)$, $Q\phi(x) = 2\pi i x\phi(x)$, $E\phi(x) = 2\pi i\phi(x)$. The dual representation on $\mathcal{S}'$ is defined by $\langle XT, \phi\rangle := \langle T, -X\phi\rangle$; note that this is consistent with the embedding $\mathcal{S} \subseteq \mathcal{S}'$, since η acts (formally) skew-adjointly.

The corresponding real Heisenberg Lie group N is the corresponding central extension of $\mathbb{R}^2$, and can be defined as triples (u, v, e) with group structure $(u, v, e) * (u', v', e') = (u + u', v + v', e + e' + (uv' - u'v)/2)$. The corresponding exponential map is then $\exp(pP + qQ + tE) = (p, q, t)$. We have $[\exp(X), \exp(Y)] = \exp([X, Y])$. The corresponding Schrödinger unitary representation of N on $\mathcal{S}$ (and even on $L^2(\mathbb{R})$) is given by

$$\exp(pP)f(x) = \cdot f(x - p)$$
$$\exp(qQ)f(x) = e^{2\pi i q x} f(x)$$
$$\exp(tE)f(x) = e^{2\pi i t} f(x).$$

This can be considered as the exponential of the infinitesimal representation of η defined above; in particular, $\frac{d}{dt}\exp(tX)f(x)|_{t=0} = iXf(x)$ for $X \in \eta$ and $f \in \mathcal{S}$. The dual representation on $\mathcal{S}'$ can be defined by $\langle XT, g\rangle := \langle T, X^{-1}g\rangle$.

Let N_m be the finite subgroup of N generated by $U_m := \exp(\frac{Q}{m})$ and $V_m := \exp(\frac{P}{m})$. This group is presented by the relations $[U_m, [U_m, V_m]] = [V_m, [U_m, V_m]] = [U_m, V_m]^{m^2} = 1$ expressing that $[U, V]$ is central and of order m^2.

0.4 Cyclic Schrödinger representations

Let $\xi : \mathbb{R} \to \mathbb{C}$ be the character $\xi(x) = e^{2\pi ix}$. Define a finite dimensional "cyclic" unitary representation of N_m on the space $\mathcal{S}'_m := C^*_{m^2}$ of complex-valued maps on $C_{m^2} := \mathbb{Z}/m^2\mathbb{Z}$ by

$$U_m f(k) := q^k f(k)$$
$$V_m f(k) := f(k-1),$$

where $q := \xi(\frac{1}{m^2})$. So $[U, V]$ acts centrally as multiplication by q.

We will show that the infinitesimal Schrödinger representation of η can be obtained as an ultralimit of these S'_m.

Embed $\mathcal{S}'_m$ into $\mathcal{S}'$ as the subspace of m-periodic linear combinations of δ functions supported at $\frac{1}{m}\mathbb{Z}$; namely, let $\text{Ш}_m := \sum_{l\in\mathbb{Z}} \delta_{ml}$ be the Dirac comb, consider the shifts by convolutions with δ functions $\delta_{\frac{k}{m}} * \text{Ш}_m = \sum_{l\in\mathbb{Z}} \delta_{ml+\frac{k}{m}}$, and define $\theta_m : \mathcal{S}'_m \to \mathcal{S}'$ by

$$\theta_m(f) := \sum_{0\le k<m^2} f(k)(\delta_{\frac{k}{m}} * \text{Ш}_m).$$

This is an embedding of representations of N_m.

Lemma 0.3. *For any non-principal ultrafilter $\mathcal{U}$, we have $\lim_{m\to\mathcal{U}} \theta_m(\mathcal{S}'_m) = \mathcal{S}'$.*

Proof. Let $T \in \mathcal{S}'$. By [Sch66, Chap.VI Sect.4], T is the limit of a sequence of smooth functions with compact support, say $T = \lim_{m\to\infty} g_m$. Define approximations $f_m \in \mathcal{S}'_m$ to g_m by sampling around 0, $f_m(k) := \frac{1}{m} g_m(\frac{k}{m})$ for $\left\lfloor \frac{-m^2}{2} \right\rfloor \le k < \left\lfloor \frac{m^2}{2} \right\rfloor$. Then for any $\phi \in \mathcal{S}$, we have $\langle g_m - \theta_m(f_m), \phi\rangle \to_{m\to\infty} 0$. So $\lim_{m\to\infty} \theta_m(f_m) = T$. Hence $\{f_m\}_m$ is bounded, and the ultralimit along any non-principal ultrafilter is T. □

Define the discrete Fourier transform on S'_m by

$$\mathcal{F}(f)(p) = \frac{1}{m} \sum_{0\le k<m^2} f(k)\xi(\frac{pk}{m^2}).$$

Then the embeddings preserve $\mathcal{F}$, i.e. $\mathcal{F}(\theta_m(f)) = \theta_m(\mathcal{F}f)$ where $\mathcal{F}$ on $\mathcal{S}'$ is the Fourier transform defined by $\langle \mathcal{F}T, \mathcal{F}\phi\rangle = \langle T, \phi\rangle$, where $\mathcal{F}$ on $\mathcal{S}$ is the usual Fourier transform defined by $(\mathcal{F}f)(p) = \int_{\mathbb{R}} f(x)\xi(px)dx$.

Indeed, $\mathcal{F}\,\mathrm{III}_1 = \mathrm{III}_1$ by Poisson summation, and similarly $\mathcal{F}\,\mathrm{III}_m = \frac{1}{m}\,\mathrm{III}_{\frac{1}{m}}$. Hence

$$\begin{aligned}
\mathcal{F}\theta_m(f) &= \sum_{0\le k<m^2} f(k)\mathcal{F}(\delta_{\frac{k}{m}} * \mathrm{III}_m) \\
&= \sum_{0\le k<m^2} f(k)\left(\left(\mathcal{F}\delta_{\frac{k}{m}}\right)\cdot(\mathcal{F}\,\mathrm{III}_m)\right) \\
&= \sum_{0\le k<m^2} f(k)\left(\left(\xi\circ(\frac{k}{m}\cdot)\right)\cdot\left(\frac{1}{m}\,\mathrm{III}_{\frac{1}{m}}\right)\right) \\
&= \frac{1}{m}\sum_{0\le k<m^2} f(k)\left(\left(\xi\circ(\frac{k}{m}\cdot)\right)\cdot\left(\sum_{0\le l<m^2}(\delta_{\frac{l}{m}} * \mathrm{III}_m)\right)\right) \\
&= \sum_{0\le l<m^2}\frac{1}{m}\sum_{0\le k<m^2} f(k)\left(\left(\xi\circ(\frac{k}{m}\cdot)\right)\cdot\left(\delta_{\frac{l}{m}} * \mathrm{III}_m\right)\right) \\
&= \sum_{0\le l<m^2}\frac{1}{m}\sum_{0\le k<m^2} f(k)\left(\xi(\frac{lk}{m^2})\cdot\left(\delta_{\frac{l}{m}} * \mathrm{III}_m\right)\right) \\
&= \sum_{0\le l<m^2}\mathcal{F}(f)(l)\cdot\left(\delta_{\frac{l}{m}} * \mathrm{III}_m\right) \\
&= \theta_m(\mathcal{F}f).
\end{aligned}$$

$\mathcal{F}$ is uniformly continuous on $\mathcal{S}'$, since $\|\mathcal{F}T\|_\phi = \|T\|_{\mathcal{F}^{-1}\phi}$. So it follows that $\mathcal{F}$ commutes with the ultralimit.

Define on $\mathcal{S}'_m$ approximations to the derivatives of $\exp(qQ)$ and $\exp(pP)$,

$$Q := \frac{m}{2}(U_m - U_m^{-1})$$

$$P := \frac{m}{2}(V_m - V_m^{-1}).$$

Lemma 0.4. *Q and P are provably bounded on the $\mathcal{S}'_m$, and they induce operators on the ultralimit $\mathcal{S}'$ defined by*

(i) $QT = (2\pi ix)\cdot T$;

(ii) $PT = -\frac{d}{dx}T$.

Proof. (i) It suffices to show that if $T = \lim_{m\to\mathcal{U}} f_m$ with $\{f_m\}_m$ bounded, then $\{Qf_m\}_m$ is bounded and $\lim_{m\to\mathcal{U}} Qf_m = (2\pi ix)\cdot T$, i.e. if $\phi = \phi(x) \in \mathcal{S}$, $\lim_{m\to\mathcal{U}}\langle Qf_m,\phi\rangle = \lim_{m\to\mathcal{U}}\langle f_m, -2\pi ix\phi\rangle$ for all $\phi\in\mathcal{S}$.

From the definition of Q on $\mathcal{S}'_m$, we have $\langle Qf_m,\phi\rangle = \langle f_m, -mi\sin(2\pi x/m)\phi\rangle$. Now $\lim_{m\to\infty} -mi\sin(2\pi x/m)\phi = -2\pi ix\phi$ in $\mathcal{S}$, since $\lim_{m\to\infty} -mi\sin(2\pi x/m) = -2\pi ix$ in $\mathcal{E}$ and multiplication is hypocontinuous. So by hypocontinuity of the pairing, $\lim_{m\to\infty}(\langle Qf_m,\phi\rangle - \langle f_m, -2\pi ix\phi\rangle) = 0$. This implies provable boundedness and that the ultralimit is as required.

(ii) One can check by direct calculation that on each $\mathcal{S}'_m$ we have $V_m = \mathcal{F}^{-1}\circ U_m\circ\mathcal{F}$, and hence by linearity $P = (\mathcal{F}^{-1}\circ Q\circ\mathcal{F})$.

Meanwhile, on $\mathcal{S}'$ we have $\frac{d}{dx} = \mathcal{F}^{-1}\circ(-2\pi ix\cdot)\circ\mathcal{F}$.

$\mathcal{F}$ is provably bounded since $\langle\mathcal{F}f_m,\phi\rangle = \langle f_m, \mathcal{F}^{-1}\phi\rangle$, and Q is provably bounded by (i).

So we conclude by Lemma 0.2.

□

0.5 Weil representations

$\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{Sp}(2,\mathbb{R})$ acts by automorphisms on η and on N, acting trivially on the centre and via the canonical action on $\mathbb{R}^2$. We will write the action in exponential notation, X^g.

Via the Stone-von Neumann theorem, this induces a projective unitary representation on S, which turns out to yield a "metaplectic" unitary representation of the double cover $\mathrm{Mp}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{S}$, such that $X^g(\phi) = (\widetilde{g} \circ X \circ \widetilde{g}^{-1})(\phi)$ for $\widetilde{g} \in \mathrm{Mp}_2(\mathbb{R})$, g the image in $\mathrm{SL}_2(\mathbb{R})$, $X \in \eta$, and $\phi \in \mathcal{S}$. The dual representation on $\mathcal{S}'$ is then defined by demanding formal unitarity, $\langle \widetilde{g}T, \widetilde{g}\phi \rangle := \langle T, \phi \rangle$.

$\mathrm{SL}_2(\mathbb{R})$ is generated by $F := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $G_b := \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ for $b \in \mathbb{R}$, so to define the metaplectic representation it suffices to define it on the lifts of these generators to $\mathrm{Mp}_2(\mathbb{R})$. Following [Fol89, p.179], we have for $\phi \in \mathcal{S}$

$$\widetilde{F}\phi(x) = \pm\xi(\frac{1}{8})\mathcal{F}\phi(x)$$

$$\widetilde{G}_b\phi(x) = \pm\xi(-b\frac{x^2}{2}) \cdot \phi(x),$$

where the signs depend on the choice of lift.

TODO: could try to be more explicit? Need a nice way to understand the group structure of $\mathrm{Mp}_2(\mathbb{R})$...

We have seen that $\mathcal{F}$, and hence the action of $\widetilde{F}$ on $\mathcal{S}'$, restricts to an operator on each $\mathcal{S}'_m$. The action of $\widetilde{G}_b$ on $\mathcal{S}'$ is by multiplication by $\pm\xi(b\frac{x^2}{2})$. For $b = \frac{c}{d}$ a rational number expressed in lowest terms, this restricts via θ_m to a partial operator $\widetilde{\mathcal{G}}_{m,b}$ on $\mathcal{S}'_m$ if d divides m^2, with domain $D_{m,b}$ consisting of functions supported on $d\mathbb{Z}/m^2\mathbb{Z} \subseteq \mathbb{Z}/m^2\mathbb{Z}$; so $\theta_m(D_{m,d}) \subseteq \theta_m(\mathcal{S}'_m)$ consists of m-periodic linear combinations of δ functions supported on $\frac{d}{m}\mathbb{Z}$.

Explicitly, $\widetilde{\mathcal{G}}_{m,b}(f)(k) := \pm\xi(b\frac{k^2}{2m^2}) \cdot f(k)$.

Call an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ <u>ultradivisible</u> if for all $d \in \mathbb{N}$, $d\mathbb{N} \in \mathcal{U}$.

Lemma 0.5. *For $\mathcal{U}$ ultradivisible, $\lim_{m\to\mathcal{U}} \theta_m(D_{m,b}) = \mathcal{S}'$.*

Proof. If $d|m$,

$$\theta_m(D_{m,d}) \supseteq \theta_m(\mathcal{S}'_{m/d}),$$

so the result follows from ultradivisibility and Lemma 0.3. □

So for $b \in \mathbb{Q}$, we could define $\widetilde{G}_b$ to be interpreted on $\mathcal{S}'_m$ as $\widetilde{\mathcal{G}}_{m,b}$, and $\widetilde{G}_b$ would then commute with the ultralimit along an ultradivisible ultrafilter. But for $b \in \mathbb{R}$, we must simultaneously approximate b by rationals.

So for $b \in \mathbb{R}$, given $m \in \mathbb{N}$, write $m = e^2 f$ where $e \in \mathbb{N}$ and f is a squarefree integer, and define $b_m := \frac{\lfloor be \rfloor}{e}$. So for $\mathcal{U}$ ultradivisible, $\lim_{m\to\mathcal{U}} b_m = b$. Then interpret $\widetilde{G}_b$ on $\mathcal{S}'_m$ as $\widetilde{\mathcal{G}}_{m,b_m}$.

Lemma 0.6. *Let $b \in \mathbb{R}$. Then $\widetilde{G}_b$ is provably bounded on the $\mathcal{S}'_m$, and for $\mathcal{U}$ ultradivisible, $\lim_{m\to\mathcal{U}}$ preserves $\widetilde{G}_b$.*

Proof. First we prove provable boundedness. So suppose $f_m \in D_{m,b_m}$ are bounded, let $\phi \in \mathcal{S}$, and let $\epsilon > 0$. Then by hypocontinuity of the pairing, for all but finitely many values of b_m, $\left|\left\langle \widetilde{G}_{b_m} f_m, \phi \right\rangle - \left\langle f_m, \widetilde{G}_{-b}\phi \right\rangle\right| =$

$|\left\langle f_m, \widetilde{G}_{-b_m}\phi - \widetilde{G}_{-b}\phi \right\rangle| = |\left\langle f_m, \pm\xi((b-b_m)\frac{x^2}{2})\phi(x) \right\rangle| < \epsilon$. So since $\left\langle f_m, \widetilde{G}_{-b}\phi \right\rangle$ is bounded, also $\left\langle \widetilde{G}_{b_m} f_m, \phi \right\rangle$ is bounded.

Now we consider the ultralimit.

If $e^2|m$, we have $D_{m,b_m} = \mathcal{S}'_e$. So as in Lemma 0.5, $\lim_{m\to\mathcal{U}} \theta_m(D_{m,b_m}) = \mathcal{S}'$.

So let $T \in \mathcal{S}'$; then we may write $T = \lim_{m\to\mathcal{U}} T_m$ with $T_m = \theta_m(f_m) \in \theta_m(D_{m,b_m})$ and bounded.

Now since $\lim_{m\to\mathcal{U}} b_m = b$, $\lim_{m\to\mathcal{U}} \xi(b_m \frac{x^2}{2}) = \xi(b\frac{x^2}{2})$ in $\mathcal{E}$.

Then by hypocontinuity of multiplication,

$$\lim_{m\to\mathcal{U}} \theta_m(\widetilde{G}_b f_m) = \lim_{m\to\mathcal{U}} \pm\xi(b_m \frac{x^2}{2}) \cdot T_m = \pm\xi(b\frac{x^2}{2}) \cdot T = \widetilde{G}_b T,$$

as required. □

We have already seen the analogous result for $\mathcal{F}$, and hence for $\widetilde{F}$. Now for $\widetilde{h} \in \mathrm{Mp}_2(\mathbb{R})$, we can write $\widetilde{h}$ as a product of instances of $\widetilde{F}$ and $\widetilde{G}_b$, and define its interpretation on $\mathcal{S}'_m$ as the corresponding composition. So by the provable boundedness and Lemma 0.2, $\lim_{m\to\mathcal{U}}$ preserves $\widetilde{h}$. So we conclude:

Theorem 0.7. *Equipping $\mathcal{S}'_m$ with the interpretations described above of P, Q, and $\widetilde{h} \in \mathrm{Mp}_2(\mathbb{R})$, the ultralimit $\lim_{m\to\mathcal{U}} \mathcal{S}'_m$ along an ultradivisible ultrafilter $\mathcal{U}$ via the embeddings θ_m is $\mathcal{S}'$ with the standard Schrödinger and Weil representations.*

References

[Fol89] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[Sch66] Laurent Schwartz. *Théorie des distributions*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris, 1966.

[Zil16] Boris Zilber. The semantics of the canonical commutation relation. 2016.