Let G be a topological group.

Def a) Let X be a compact Hausdorff space. A flow (or a G-flow) is a continuous action $G \times X \to X$ of G on X.
Notation: $G \curvearrowright X$; X is a G-flow

b) A G-ambit is a G-flow X with a distinguished point $x_0 \in X$, whose orbit is dense in X.
Notation: (X, x_0) is a G-ambit

c) A G-flow X is minimal if every orbit is dense

d) Let (X, x_0) and (Y, y_0) be G-ambits.
A continuous map $\pi : X \to Y$ is a homomorphism if:

- $\forall x \in X \ \forall g \in G \quad \pi(gx) = g\pi(x)$
  $\pi(x_0) = y_0$

$\to$ Note that homomorphism has to be onto

Theorem Given a topological group G, there is a G-ambit $(S(G), s_0)$ with the following property: For every G-ambit (X, x_0) there is a homomorphism of $(S(G), s_0)$ onto (X, x_0). Moreover, $(S(G), s_0)$ is uniquely determined up to isomorphism by this property.

uniqueness: (X,x_0), (Y,y_0) - univ. G-ambits
We have homomorphisms
$\pi_1 : (X, x_0) \to (Y, y_0)$, $\pi_2 : (Y, y_0) \to (X, x_0)$
Then $\pi_2 \circ \pi_1$ is onto and id. on the dense set Gx_0. Since $\pi_2 \circ \pi_1$ is continuous, we have $\pi_2 \circ \pi_1 = id_X$. Similarly $\pi_1 \circ \pi_2 = id_Y$. So π_1 and π_2 are isomorphisms.

We now describe in a couple of ways the G-ambit, where G is a topological group.

0) G - discrete group, e - the identity of G
Then $(\beta G, e)$ is the universal G-ambit
$(\beta G$ = the space of all ultrafilters on G,

e - the ultrafilter containing $\{e\}$,
also denotes here

$g\mathcal{U} = \{gX : X \in \mathcal{U}\}$ ← the action
$\mathcal{U} \in \beta G$

Recall:

- An ultrafilter $\mathcal{U}$ is a collection of subsets of G s.t.

→ $\forall X, Y \in \mathcal{U}$ $X \cap Y \in \mathcal{U}$
→ empty set is not in $\mathcal{U}$
→ $\forall X, Y$ if $X \subseteq Y$ and $X \in \mathcal{U}$, then $Y \in \mathcal{U}$
→ $\forall X \subseteq G$ $X \in \mathcal{U}$ or $(G \setminus X) \in \mathcal{U}$

- βG is compact in the topology given by the basic of open sets:

$$S_X = \{\mathcal{U} \in \beta G : X \in \mathcal{U}\}, \quad \text{where } X \subseteq G.$$

1) Now let G be a topological group
G_d - group G, but taken with the discrete topology

Consider $\beta(G_d)$.

For $u \in \beta(G_d)$ we consider the filter u^* of open sets in G generated by

$$\{VA : A \in u, V \in \mathcal{N}\},$$

where $\mathcal{N}$ is the nbhd basis of the identity.

For $u, v \in \beta(G_d)$, we set

$$u \sim v \quad \text{iff} \quad u^* = v^*$$

Then the quotient space

$$S(G) = \beta(G_d)/\sim$$

together with the distinguished point e, is the greatest ambit of G.

2) G - topological group in the language

$L = \{ \cdot , {}^{-1}, e, (U : U \text{ - open in } G) \}$

$\uparrow$ unary predicate for U

(We may want to add constants $g \in G$ to L.)

Let $G^* \succ G$ be a monster model

(κ - saturated and strongly κ - homogeneous)

$\kappa > |L|, |G|$

Denote $U^* = U(G^*)$

- Let μ be the group of infinitesimals:

$\mu = \bigcap \{ U^* : U$ - open nbhd of the $e \in G \}$

Define $\sim$ on G^* by

$$a \sim b \iff ab^{-1} \in \mu$$

- Let also $\equiv_G$ be the relation of having the same type $/G$

- Finally, let

$$E_{\mathcal{N}} := \sim \circ \equiv_G = \underset{\text{eq. relation}}{\equiv_G} \circ \underset{\text{not eq. relation}}{\sim}$$

(so $x \, E_{\mathcal{N}} \, y \iff \exists z \;\; x \sim z$ and $z \equiv_G y$)

Clearly

→ $E_{\mathcal{N}}$ is coarser than $\sim$ and than $\equiv_G$

→ If some F is coarser than both $\sim$ and $\equiv_G$, then $E_{\mathcal{N}}$ is finer than F.

- Consider $G^* / E_{\mathcal{N}}$ equipped with the logic topology

(i.e. closed subsets are those subsets whose preimages by the quotient map are type definable subsets of G^*)

↑ i.e. intersections of definable sets

→ The G^* is compact in the topology given by type definable subsets of G^*

So $G^*/E_{\mathcal{N}}$ is compact

→ Each of $\sim$ and $\equiv_G$ are type definable, so is $E_{\mathcal{N}}$. Therefore $E_{\mathcal{N}}$ is closed in G^*

So $G^*/E_{\mathcal{N}}$ is Hausdorff (G^* may not be Hausdorff)

→ The action:

$$g(a/E_\mu) = ga/E_\mu$$

Theorem

The $(G^*/E_\mu, e/E_\mu)$ is the universal G-ambit.

Proof

- The $G \cdot e/E_\mu = G/E_\mu$ is dense in G^*/E_μ, since G is already dense in G^* (note that in fact the topology basis in G^* is given by definable sets).

- Let (X, x_0) be a G-ambit. Then $f : G \to X$ given by $f(g) = gx_0$ is continuous (and so definable).
preimages of closed sets are type definable

- f extends to a G-definable $f^* : G^* \to X$:

  Let $c \in G^*$. Let $p(y) = tp(c/G)$. Then $\bigcap_{\varphi \in p} \overline{f(\varphi(G))}$ is a singleton $\{x_c\}$. Let $f^*(c) = x_c$.

- We have to show that f^* factors through E_μ. It suffices to show that each of $a \sim b$ and $a \equiv_G b$ implies $f^*(a) = f^*(b)$.

→ if $a \equiv_G b$, it follows from the def of f^*

→ suppose that $a \sim b$ and towards contradiction $f^*(a) \neq f^*(b)$. This gives (by compactness of X) $\varphi \in tp(a/G)$ and $\psi \in tp(b/G)$ s.t.

$$\overline{f(\varphi(G))} \cap \overline{f(\psi(G))} = \emptyset .$$

By compactness of X and continuity $G \curvearrowright X$ there is an open nbhd U of e s.t.

$$U \underbrace{\varphi(G)\, x_0}_{f(\varphi(G))} \cap \psi(G)\, x_0 = \emptyset .$$

Hence $U\varphi(G) \cap \psi(G) = \emptyset$, which implies

$$\psi(G)\, \varphi(G)^{-1} \cap U = \emptyset .$$

But this contradicts $ab^{-1} \in \psi(G^*)\varphi(G^*)^{-1} \cap \mu$.

- We also have $g f^*(a) = f^*(ga)$ from the definitions, which implies $g f^*(a/E_\mu) = f^*(ga/E_\mu)$. ∎

From that we get the universal right G-ambit

Take $E_\mu^r = E_\mu^{-1} = \equiv_G \circ \sim_r = \sim_r \circ \equiv_G$

where $a \sim_r b$ iff $a^{-1} b \in \mu$

action: $(a/E_\mu^r)\, g = (ag)/E_\mu^r$

Now let M be an arbitrary first order structure in a language L, and let $G = Aut(M)$ be equipped with the pointwise convergence topology.

Let $\mathcal{M}$ be the structure consisting of two disjoint sorts G and M with predicates for <u>all</u> subsets of finite cartesian products of sorts. We call this language <u>full</u>.

Let $\mathcal{M}^* = (G^*, M^*, \ldots) \succ \mathcal{M}$ be a monster model (of $Th(\mathcal{M})$).

Enumerate M as $\overline{m}$. Define

$$\Sigma^{\mathcal{M}} := \{ tp^{full}(\sigma(\overline{m})) : \sigma \in G^* \}$$

- G acts on $\Sigma^{\mathcal{M}}$ via

$$tp^{full}(\sigma(\overline{m})) \cdot g = tp^{full}(\sigma(g(\overline{m})))$$

- basic clopen sets of $\Sigma^{\mathcal{M}}$:

$[\varphi(\bar{x})] =: \{ p \in \Sigma^{\mathcal{M}} : \varphi(\bar{x}) \in p \}$, $\varphi(\bar{x})$ - formula without parameters in the full language

Theorem

$(\Sigma^{\mathcal{M}}, tp^{full}(\bar{m}))$ is the universal right G-ambit

And if we consider $g \cdot tp^{full}(\sigma(\bar{m})) = tp^{full}(\sigma(g^{-1}(\bar{m}))$,
we get the universal (left) G-ambit.

$\to$ It is a right G-ambit

$\to$ It is universal, since it is isomorphic
to G^*/E_ν^r $\leftarrow$ where L is taken to be L^{full}

Let $F: G^* \to \Sigma^{\mathcal{M}}$ be given by
$F(\sigma) = tp^{full}(\sigma(\bar{m}))$

Claim

$F(\sigma) = F(\tau) \iff \sigma E_\nu^r \tau$

Proof

$(\Rightarrow)$ Assume $F(\sigma) = F(\tau)$. Then there is
$f \in Aut(\mathcal{M}^*)$ with $f(\sigma(\bar{m})) = \tau(\bar{m})$.
So $\tau^{-1} f \sigma(\bar{m}) = \bar{m}$. Since
$\nu = \{\sigma \in G^* : \sigma(\bar{m}) = \bar{m}\}$, we obtain
$\tau \sim^r f\sigma \equiv^{full}_{\mathcal{M}} \sigma$,
which means $\tau E_\nu^r \sigma$.

($\Leftarrow$) Suppose $\sigma E^r_\nu \tau$. Then there is $f \in Aut(\mathcal{M}^*)$ with $f\sigma \sim^r \tau$. This means $f\sigma(\bar{m}) = \tau(\bar{m})$. Hence $tp^{full}(\sigma(\bar{m})) = tp^{full}(\tau(\bar{m}))$.

■ Claim

- So F induces a bijection $\tilde{F}: G^*/E^r_\nu \to \Sigma^{\mathcal{M}}$.

- $\tilde{F}$ is continuous:

no parameters, in full language

$\tilde{F}^{-1}[[\varphi(\bar{x})]] = \{\sigma/E^r_\nu : \models \varphi(\sigma(\bar{m}))\}$ is closed in the logic topology

- Moreover for $\sigma \in G^*$, $g \in G$

$\tilde{F}((\sigma/E^r_\nu)\cdot g) = tp^{full}(\sigma(\bar{m}))\cdot g$

- And $\tilde{F}(e/E^r_\nu) = tp^{full}(\bar{m})$.

■

Remark

G - discrete

the isomorphism of universal G-ambits $\tilde{f}: \beta G \to \Sigma^{\mathcal{M}}$ is the unique continuous extension of the map $f: G \to \Sigma^{\mathcal{M}}$ given by $f(g) = tp^{full}(g(\bar{m}))$.