Lec 6

Recall

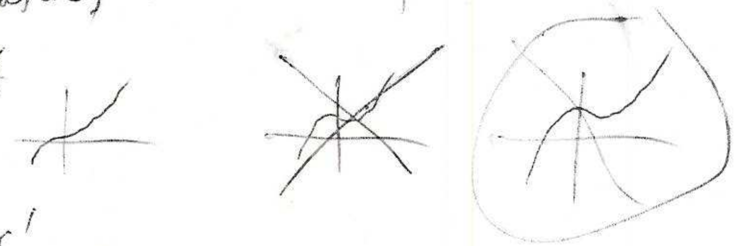
Suppose a function f is twice continuously differentiable on an interval I ← $(2,3)$, $[-4,5)$, $(0,\infty)$

i.e. f'' is defined and is continuous on I

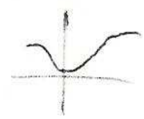
Then

- f is increasing I $\iff$ f' is ≥ 0 on I

f

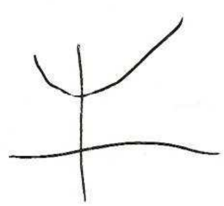


f'



Better? Yes!

- f is decreasing on I $\iff$ f' is ≤ 0 on I
- f is concave upward on I $\iff$ f'' is ≥ 0 on I



concave downwards

concave upwards

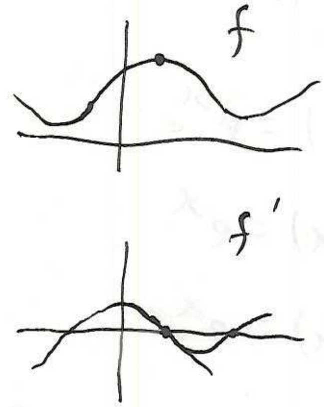
" " " downwards " " " " ≤ 0 "

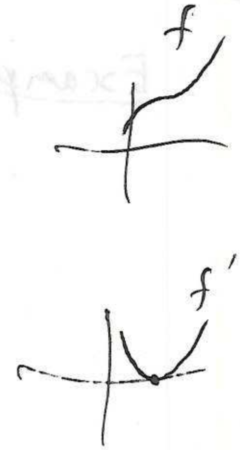
If c is in I, then

f has a local maximum at c

$\Leftrightarrow$ "f increasing on the left and decreasing on the right"

$\Leftrightarrow$ $f' \geq 0$ on some interval (a, c)
and $f' \leq 0$ on some interval (c, d)
and $f'(c) = 0$





(symmetrically for local min)

f has an inflection point at c

$\Leftrightarrow$ f changes from being concave up to concave down at c
or the other way round

$\Leftrightarrow$ $f''(c) = 0$
and the sign of f'' changes at c

$$e^y \frac{dy}{dx}$$

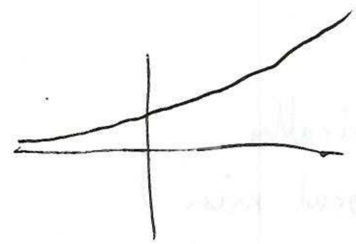
$$\text{so } \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\boxed{\text{so } \frac{d}{dx} \ln x = \frac{1}{x}}$$

③

- Examples

  - $f(x) = e^x$

    $f'(x) = e^x$

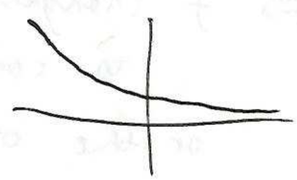
    $f''(x) = e^x$

    $e^x > 0$ for all x

    so f is increasing

    and concave up everywhere

  - $f(x) = e^{-x}$

    $f'(x) = -e^{-x}$

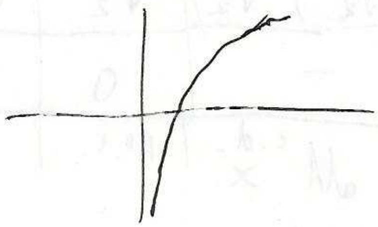
    $f''(x) = -(-e^{-x}) = e^{-x}$

    so e^{-x} is decreasing and concave up

- $f(x) = \ln x$ only defined for $x > 0$

$f'(x) = 1/x$ so f is increasing

$f''(x) = \frac{-1}{x^2}$ and concave down for all $x > 0$



- $f(x) = e^{-x^2}$

$f'(x) = -2x\,e^{-x^2}$

$f''(x) = -2e^{-x^2} + (-2x)(-2x e^{-x^2})$

$= (4x^2 - 2)e^{-x^2}$

Chain rule $\Rightarrow \frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}$

Product rule: $\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$

$f'(x) = 0 \Leftrightarrow x = 0$ (since $e^{-x^2} \neq 0$ for any x)

$f''(x) = 0 \Leftrightarrow 4x^2 - 2 = 0$

$\Leftrightarrow 4x^2 = 2$

$\Leftrightarrow x^2 = 1/2$

$\Leftrightarrow x = \pm 1/\sqrt{2}$ $(= \pm 0.71)$

signs:

x	$(-\infty, 0)$	0	$(0, \infty)$
$f'(x)$	$+$	0	$-$

$\Rightarrow$ f has local max at 0

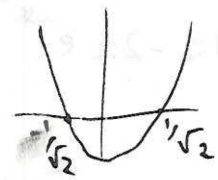
x	$(-\infty, -1/\sqrt{2})$	$\frac{-1}{\sqrt{2}}$	$\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$(1/\sqrt{2}, \infty)$
$f''(x)$	$+$	0	$-$	0	$+$
$f(x)$	c.u.	p.o.i	c.d.	p.o.i.	c.u.

($e^{-x^2} > 0$ for all x

so the sign of $f''(x) = (4x^2-2)e^{-x^2}$

is the same as the sign of $4x^2-2$

for all x)



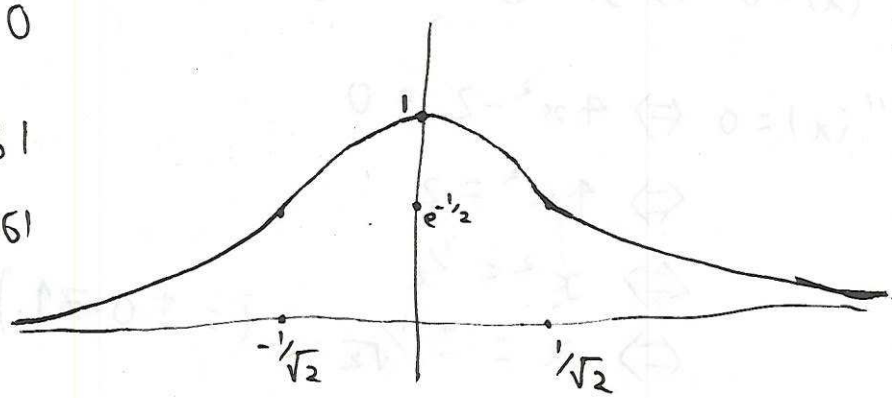
$$\lim_{x \to +\infty} e^{-x^2} = \lim_{t \to -\infty} e^t = 0$$

$$\lim_{x \to -\infty} e^{-x^2} = 0$$

$f(-1/\sqrt{2}) = e^{-1/2} = 0.61$

$f(1/\sqrt{2}) = e^{-1/2} = 0.61$

$f(0) = e^0 = 1$



$f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$ "Gaussian / Normal distribution"

so f is even