Lec 9

xkcd.org

$$\log_{10} F = at + b$$

$$F = 10^{at+b}$$
$$= 10^b \, 10^{at}$$
$$= 10^b \left(e^{\ln 10}\right)^{at}$$
$$= 10^b e^{(a \ln 10)t}$$

$$F/10^b = e^{(a\ln 10)t}$$

so rescaling, $F_1 := F/10^b \quad t_1 := (a \ln 10)t$

$$F_1 = e^{t_1}$$

Example:

An object is released from rest, and experiences a constant downwards acceleration of $10\,ms^{-2}$

Quⁿ (i) what is its velocity t seconds after release

(ii) What is its height " " " "

Answer (i) $a(t) = -10$ $\quad v'(t) = a(t) \quad h'(t) = v(t)$

$$v(t) = \int a(t)\,dt$$
$$= \int -10\,dt$$
$$= -10t + c$$

$$\left(-10 = -10t^0 \quad \int -10t^0\,dt = -10\frac{t^1}{1} = -10t \right.$$

$v(0) = 0$

so $0 = v(0) = (-10)(0) + c = c$

so $c = 0$

so $v(t) = -10t$

(ii) $$h(t) = \int v(t)\,dt$$
$$= \int -10t\,dt$$
$$= -10\tfrac{1}{2}t^2 + c$$

Not enough information to find c

Revised Question (ii): suppose the object hits the ground 2 seconds after release. what's the height $h(t)$ above the ground it sees after release? (3)

<u>Answer</u>: $h(t) = \cancel{} -5t^2 + c$

$h(2) = 0$

so $0 = h(2) = (-5)(2^2) + c$

$= -20 + c$

so $c = 20$

so $h(t) = \cancel{}\ 20 - 5t^2$

<u>Generally</u>: If f is a continuous function on an interval I, then given any b in I and any d, f has a <u>unique</u> antiderivative F such that

$F(b) = d$

"If you know F' and a single value of F, then you know F"

<u>Why?</u> We saw that the antiderivatives of f are of the form $G(x) + c$

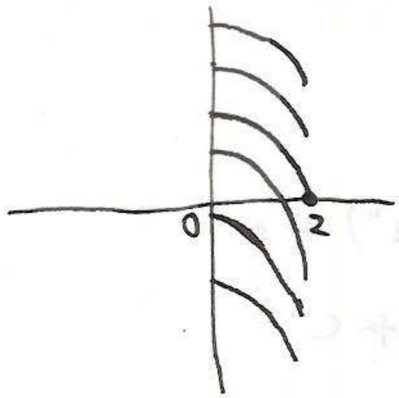
(fixed G, varying c)

For exactly one c, we have $G(b) + c = d$

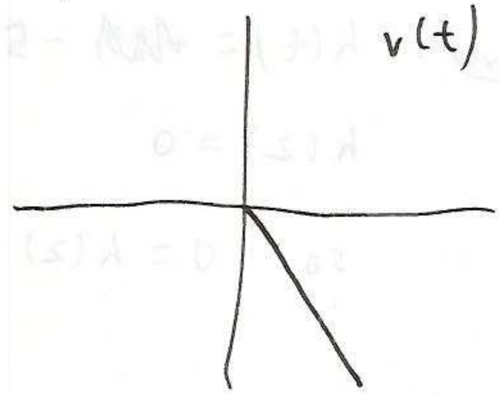
so $F(x) = G(x) + c$ for that c

Graphically:

$h(t) = -5t^2 + c \qquad h'(t) = v(t)$



$v(t) = -10t$



"Just need to fix one point on the graph to fix the whole function"

Linearity of Integration

Recall:
$$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$
$$\frac{d}{dx}\left(\lambda f(x)\right) = \lambda\frac{d}{dx}f(x) \qquad (\lambda \text{ any real})$$

"$\frac{d}{dx}$ is a linear operator"

(note these fit:
$$\frac{d}{dx}\left(2f(x)\right) = \frac{d}{dx}\left(f(x)+f(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}f(x) = 2\frac{d}{dx}f(x))$$

(note we deduce
$$\frac{d}{dx}\left(f(x)-g(x)\right) = \frac{d}{dx}\left(f(x)+(-1)g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}\left((-1)g(x)\right) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x))$$

How about for integration?
$$\int \left(f(x)+g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx \quad ?$$

Yes: $\frac{d}{dx}\left(\int f(x)\,dx + \int g(x)\,dx\right)$

$$= \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx$$

$$= f(x) \quad + \quad g(x)$$

Similarly $\int \lambda f(x)\,dx = \lambda \int f(x)\,dx$ any real λ.

e.g. $\int (3e^x + 7x^{3/2})\,dx$

$$= \int 3e^x\,dx + \int 7x^{3/2}\,dx$$

$$= 3\int e^x\,dx + 7\int x^{3/2}\,dx$$

$$= 3e^x + 7\tfrac{2}{5}x^{5/2} + c$$

Note: We only need one "$+c$"!

Could have written $3(e^x + c_1) + 7(\frac{2}{5}x^{5/2} + c_2)$

But as we vary c_1 and c_2,

$(3c_1 + 7c_2)$ ranges through $\mathbb{R}$

so we can just call it c

State of Play: We can integrate:

- x^α
- e^x
- linear combinations of these

(e.g. polynomials

$$\int (7x^5 + \sqrt{2}x^2 + \pi)\,dx = \frac{7}{6}x^6 + \frac{\sqrt{2}}{3}x^3 + \pi x + c$$

How about

- $\int e^{x^2}\,dx$
- $\int x e^{x^2}\,dx$
- $\int x^2 e^{x^2}\,dx$