Lec 10

Math Help Centre

Mon - Fri 2:30 HH/104

Substitution

Recall: chain rule: $\frac{d}{dx} f(g(x)) = g'(x) f'(g(x))$

e.g. $y = e^{x^2}$

$\frac{dy}{dx} = 2x e^{x^2}$ $(g(x) = x^2,\ f(x) = e^x$

$y = e^{x^2} = f(g(x)))$

in other words,

if we "make the substitution" $u = x^2$

then $\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$ $($ $y = e^u$

$\frac{dy}{du} = e^u = e^{x^2})$

In general: If $y = f(g(x))$

if we substitute $u = g(x)$

so $y = f(u)$

then the chain rule says $\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$

in other words: $\frac{d}{dx} f(g(x)) = g'(x) \cdot f'(u)$

$= g'(x) \cdot f'(g(x))$

Integration by Substitution:

Examples:

$$\int e^{5x}\,dx = ?$$

$$\frac{d}{dx} e^{5x} = \frac{du}{dx}\frac{d}{du} e^{u} \qquad u = 5x$$

$$= 5 \cdot e^{u}$$

$$= 5 e^{5x}$$

So $\frac{d}{dx}\frac{1}{5}e^{5x} = \frac{1}{5}\frac{d}{dx}e^{5x} = \frac{1}{5}5e^{5x} = e^{5x}$

so $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + c$

$$\int 3x^2 (x^3+7)^{37}\,dx = \frac{1}{38}(x^3+7)^{38} + c$$

$$\frac{d}{dx}(x^3+7)^{37} = \frac{du}{dx}\cdot\frac{d}{du}u^{37} \qquad u = x^3+7$$

$$\frac{du}{dx} = 3x^2$$

$$= 3x^2 (37) u^{36}$$

$$= 3x^2 (37)(x^3+7)^{36}$$

$$\frac{d}{dx}\frac{1}{38}(x^3+7)^{38} = \frac{3x^2 (38)(x^3+7)^{37}}{38}$$

$$= 3x^2 (x^3+7)^{37}$$

Substitution rule: If $u = g(x)$

then $$\boxed{\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du}$$

(f cont's function, g is differentiable)

Proof: Say $\frac{d}{du}F(u) = f(u)$ (i.e. $F' = f$)

$$\frac{d}{dx}F(u) = \frac{d}{dx}F(g(x)) = g'(x)\,F'(g(x))$$

$$= g'(x)\,f(g(x))$$

$$= \frac{du}{dx}\,f(u)$$

$\square$

Example $\int e^x e^{e^x}\,dx = \int \frac{du}{dx} e^u\,dx$ $\qquad u = e^x$

$= \int e^u \frac{du}{dx}\,dx$ $\qquad \frac{du}{dx} = e^x$

$= \int e^u\,du$ (by the substitution rule

$= e^u + c$

$= e^{e^x} + c$

$\int x e^{x^2}\,dx = \int x e^u\,dx$ $\qquad u = x^2$

$= \frac{1}{2}\int 2x e^u\,dx$ $\qquad \frac{du}{dx} = 2x$

$= \frac{1}{2}\int \frac{du}{dx} e^u\,dx$

$= \frac{1}{2}\int e^u\,du$

$= \frac{1}{2}e^u + c$

$= \frac{1}{2}e^{x^2} + c$

$\int e^{x^2}\,dx = \int e^u\,dx$ $\qquad u = x^2$

$= ?$ $\qquad \frac{du}{dx} = 2x$

No use $\left(\int e^u\,dx \neq \frac{1}{x}\int x e^u\,dx \right)$

$$\int \sqrt{x}\, e^{x\sqrt{x}}\, dx = \int u e^{xu}\, dx \qquad u = \sqrt{x}$$

$$= \int u e^{u^3}\, dx \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$$

$$u = x\sqrt{x} = x^{3/2}$$

$$\int \sqrt{x}\, e^{x\sqrt{x}}\, dx = \frac{2}{3}\int \frac{3\sqrt{x}}{2} e^{u}\, dx \qquad \frac{du}{dx} = \frac{3\sqrt{x}}{2}$$

$$= \frac{2}{3}\int e^{u} \frac{du}{dx}\, dx$$

$$= \frac{2}{3}\int e^{u}\, du$$

$$= \frac{2}{3} e^{x\sqrt{x}} + c$$

$$\int \frac{\ln x + x^2}{x}\, dx = \int \left(\frac{\ln x}{x} + x\right) dx$$

$$= \int \frac{\ln x}{x}\, dx + \int x\, dx$$

$$= \int u \frac{du}{dx}\, dx + \frac{x^2}{2} \qquad u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$= \int u\, du + \frac{x^2}{2}$$

$$= \frac{u^2}{2} + \frac{x^2}{2} + c$$

$$= \frac{(\ln x)^2}{2} + \frac{x^2}{2} + c$$

Yes : $\frac{d}{dx}\left(\int f(x)\,dx + \int g(x)\,dx\right)$

$$= \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx$$

$$= f(x) \quad + \quad g(x)$$

Similarly $\int \lambda f(x)\,dx = \lambda \int f(x)\,dx$ any ~~real~~ real λ.

e.g. $\int (3e^x + 7x^{3/2})\,dx$

$$= \int 3e^x\,dx + \int 7x^{3/2}\,dx$$

$$= 3\int e^x\,dx + 7\int x^{3/2}\,dx$$

$$= 3e^x + 7\tfrac{2}{5}x^{5/2} + c$$

<u>Note</u>: We only need one "$+c$"!

could have written $3(e^x + c_1) + 7(\frac{2}{5}x^{5/2} + c_2)$

But as we vary c_1 and c_2, $(3c_1 + 7c_2)$ ~~ranges~~ ranges through $\mathbb{R}$

so we can just call it c

<u>State of Play</u>: We can integrate:
- x^α
- e^x
- linear combinations of these (e.g. polynomials

$$\int (7x^5 + \sqrt{2}x^2 + \pi)\,dx = \frac{7}{6}x^6 + \frac{\sqrt{2}}{3}x^3 + \pi x + c$$

How about
- $\int e^{x^2}\,dx$
- $\int x e^{x^2}\,dx$
- $\int x^2 e^{x^2}\,dx$