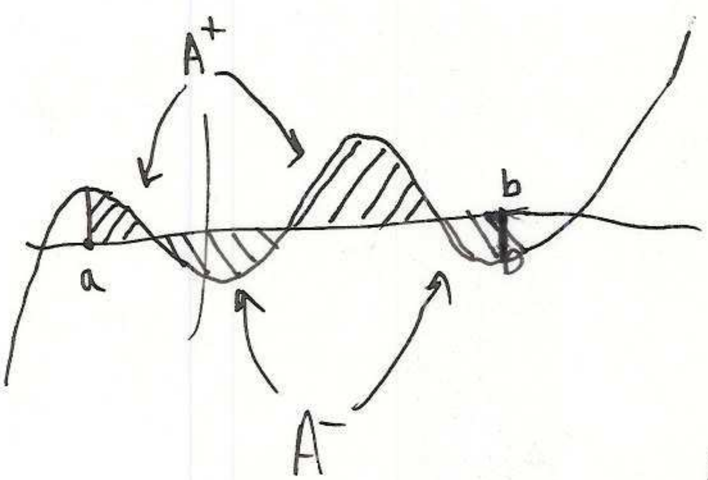
The Fundamental Theorem of Calculus

· Fix $a < b$ and a ~~ctn~~ function f continuous on $a \le x \le b$

· Def^n: The <u>signed area</u> between the graph of f and the x-axis between a and b



is $A = A^+ - A^-$

where A^+ is the area of the region(s) above the x-axis and below the graph of f between a and b

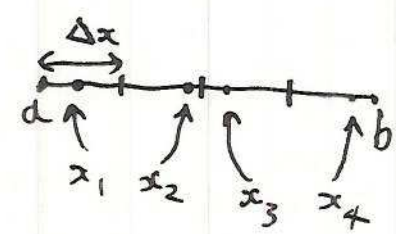
A^- area below x-axis above graph

Estimating this signed area:

For each natural number n:

· divide the interval into n equally sized subintervals so each has length $\Delta x = \frac{b-a}{n}$

$n=4$

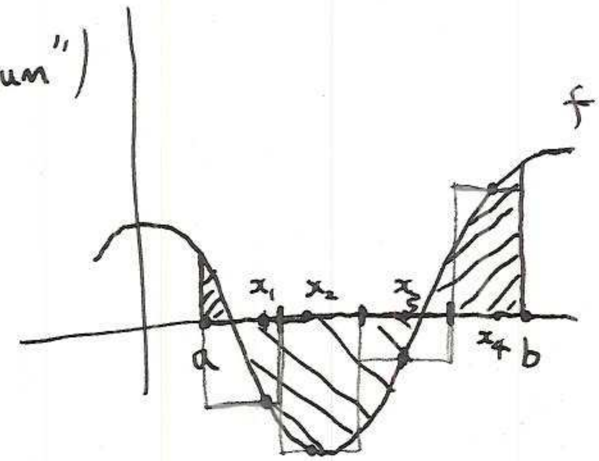


· pick points $x_1, \dots, x_n$ such that x_i lies in the ith subinterval

· Let $A_n := [\text{sum of signed areas of rectangles}]$
$$= \Delta x f(x_1) + \Delta x f(x_2) + \dots + \Delta x f(x_n)$$

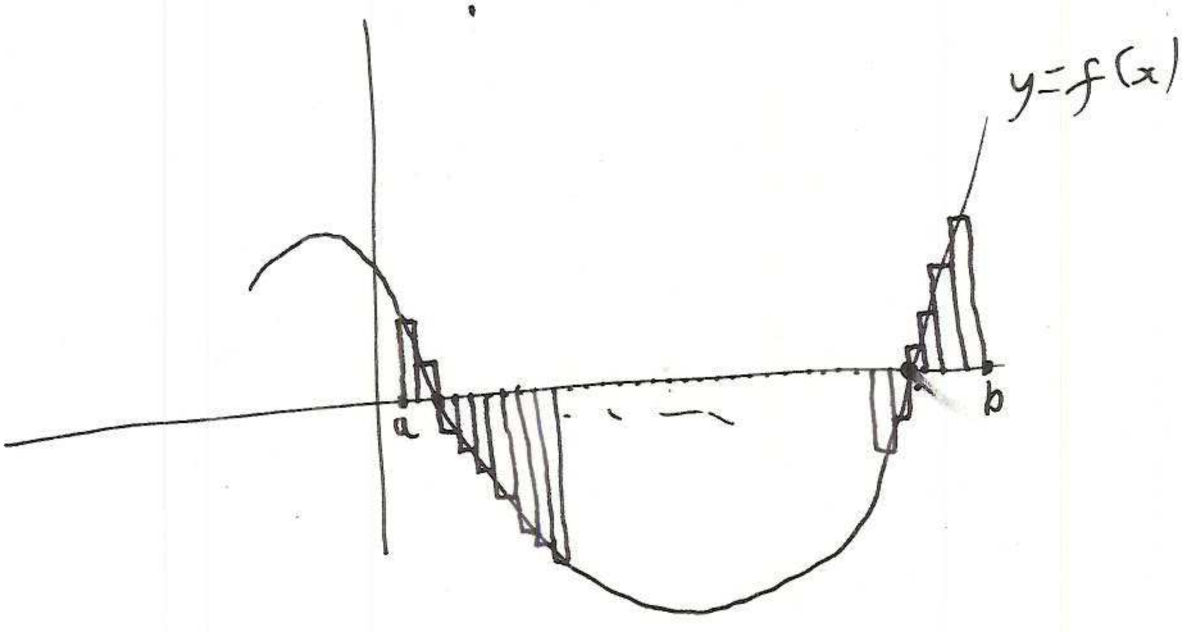
$= \Delta x \left(f(x_1) + \dots + f(x_n)\right)$

$= \Delta x \sum_{i=1}^{n} f(x_i)$ ("Riemann sum")



For large n, expect A_n to be close to the actual signed area between the graph of f and the x-axis between a and b

Signed Area of ith rectangle is $\Delta x \cdot f(x_i)$



Indeed,

<u>Fact</u>: There is a real number A such that however we pick the x_i for each n,

$$\lim_{n\to\infty} A_n = A$$

We call this number A
the definite integral of f from a to b
and write it

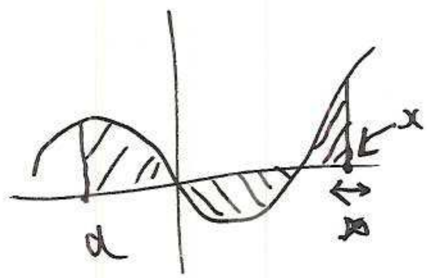
$$\int_a^b f(x)\,dx \qquad \left(= \int_a^b f(t)\,dt \right.$$

$$= \int_a^b f(\xi)\,d\xi$$

$$\left. = \int_a^b f(\text{♀})\,d\text{♀} \right)$$

If $a > b$ define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Now let $A(x) := \int_a^x f(t)\,dt$



We argued on Tuesday
that $A'(x) = f(x)$

i.e. $$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

so $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$

so: (I) any continuous function has an antiderivative

(II) if we have an antiderivative of $f(x)$ whose values we can calculate then we can calculate $\int_a^b f(x)\,dx$

i.e. we can calculate signed areas

Indeed:

<u>Theorem</u> [FTC (Fundamental Th^{n} of Calculus)]:

If $f(x)$ is continuous on an interval $a \le x \le b$ and if $F(x)$ is an antiderivative on $a \le x \le b$

then $$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

<u>Proof</u>: $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$ on $a \le x \le b$

so for some c, for all x $(a \le x \le b)$

$$\int_a^x f(t)\,dt = F(x) + c$$

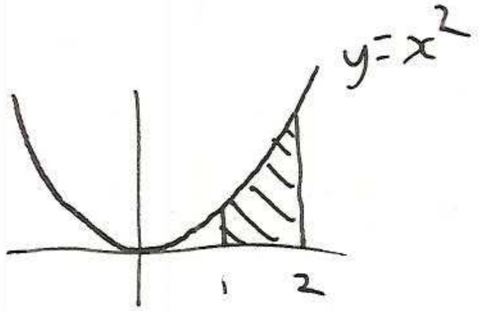
but $0 = \int_a^a f(t)\,dt = F(a) + c$

a

so ~~F(a)~~ $c = -F(a)$

so $\displaystyle\int_a^b f(t)\,dt = F(b) - F(a)$ $\square$

e.g.



$$\int_1^2 x^2\,dx = ?$$

$$\int x^2\,dx = \frac{x^3}{3} + c$$

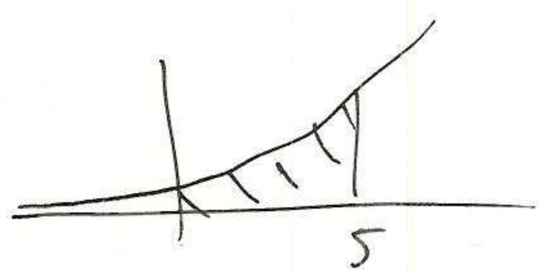
so $F(x) = \dfrac{x^3}{3}$ is an antiderivative of x^2

so by FTC, $\displaystyle\int_1^2 x^2\,dx = F(2) - F(1)$

$$= \frac{8}{3} - \frac{1}{3}$$

$$= 7/3$$

$$\int_0^5 e^{2x}\,dx = ?$$



$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$$

so $\displaystyle\int_0^5 e^{2x}\,dx = \frac{1}{2}e^{(2)(5)} - \frac{1}{2}e^{(2)(0)} = \frac{1}{2}e^{10} - \frac{1}{2}$

Notation: $\left[F(x)\right]_{x=a}^{x=b} = F(b) - F(a)$

(Abbreviated: $\left[F(x)\right]_a^b$
or $F(x)\Big]_a^b$

so FTC says

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$