Improper Integrals (§6.2)

Example: The nuclear waste product Caesium-137 radiates energy at a rate of 0.6 W per gram of Caesium-137, and has a half-life of 30 years.

How much energy will one gram release?

Solution: After t years, $2^{-t/30}$ grams remain, which radiates at a rate of

$$P(t) = (0.6)\, 2^{-t/30} \quad J/s$$

$$= (0.6)\, 2^{-t/30} \left((365)(24)(60^2)\right) \quad J/year$$

$$= (19)(10^6)\, 2^{-t/30} \quad J/year$$

$$= (19)\, 2^{-t/30} \quad MJ/year$$

So, after N years, the amount of energy in MJ which has been radiated is

$$\frac{d}{dt} e^{ct} = c e^{ct}$$

$$E(N) = \int_0^N (19)\, 2^{-t/30}\, dt$$

$$= 19 \int_0^N \left(e^{\ln 2}\right)^{-t/30} dt \qquad \int e^{ct}\, dt = \frac{1}{c} e^{ct} + C$$

$$= 19 \int_0^N e^{\left(-\frac{\ln 2}{30} t\right)} dt$$

$$= -19 \frac{30}{\ln 2} \left[e^{-\frac{\ln 2}{30} t} \right]_0^N$$

$$= -820 \left(e^{-0.023 N} - 1\right)$$

$$= 820 \left(1 - e^{-0.023 N}\right)$$

e.g. after 1 year,

$$E(1) = 820 \left(1 - e^{-0.023}\right) = 19$$

10 years

$$E(10) = 820(1 - e^{-0.23}) = 170$$

$$E(100) = 740$$

$$E(1000) = 820$$

$$\lim_{N \to \infty} E(N) = \lim_{N \to \infty} 820(1 - e^{-0.023N})$$

$$= 820(1 - \lim_{N \to \infty} e^{-0.023N})$$

$$= 820(1 - 0)$$

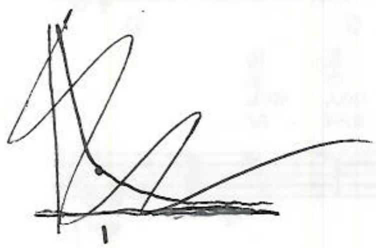
$$= 820$$

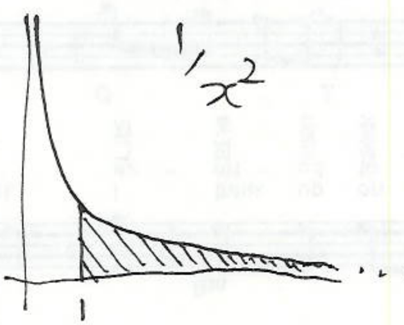
Definition: $\displaystyle\int_a^\infty f(x)\,dx = \lim_{N \to \infty} \int_a^N f(x)\,dx$

Sometimes the limit exists sometimes doesn't

e.g. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N \to \infty} \int_1^N \frac{1}{x^2}\,dx = \lim_{N \to \infty} \left[\frac{-1}{x}\right]_1^N$

$$= \lim_{N \to \infty} \left(\frac{-1}{N} - \frac{-1}{1}\right)$$



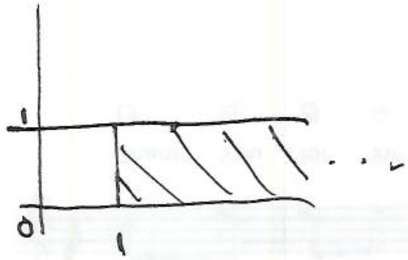


$$= \lim_{N \to \infty} \left(1 - \frac{1}{N}\right)$$

$$= 1$$

$$\int_1^\infty 1\,dx = \lim_{N \to \infty} \int_1^N 1\,dx = \lim_{N \to \infty} [x]_1^N = \lim_{N \to \infty} (N-1)$$

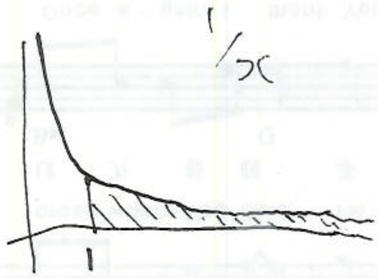
no finite limit



$$\int_1^\infty \frac{1}{x}\,dx = \lim_{N\to\infty}\int_1^N \frac{1}{x}\,dx = \lim_{N\to\infty}\left[\ln x\right]_1^N$$

$$= \lim_{N\to\infty}\left(\ln N - \ln 1\right)$$

$$= \lim_{N\to\infty} \ln N$$



no finite limit

$$\int_0^\infty e^{cx}\,dx = \lim_{N\to\infty}\int_0^N e^{cx}\,dx$$

$$= \lim_{N\to\infty}\left[\frac{1}{c}e^{cx}\right]_0^N$$

$$= \lim_{N\to\infty}\frac{1}{c}\left(e^{cN}-1\right)$$

$\cancel{\int_0^\infty}$

$\cancel{\int_1^\infty \frac{\ln x}{x^2}\,dx}$

$\cancel{\int_0^\infty x^2 e^{-x/7}\,dx}$

if $c>0$ ~~$\lim_{N\to\infty} e^{cN}$ has no finite~~

$$e^{cN} \xrightarrow[N\to\infty]{} \infty$$

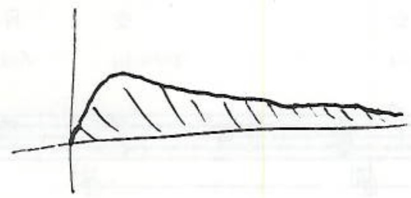
so integral has no finite limit

if $c<0$, $e^{cN} \xrightarrow[N\to\infty]{} 0$

so $\int_0^\infty e^{cx}\,dx = \cancel{M}\ \frac{-1}{c}$ e.g. $\int_0^\infty e^{-7x}\,dx = \frac{1}{7}$

(so $\int_0^{\infty} e^{-x}\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx = 1$

· $\int_0^{\infty} x e^{-x/2}\,dx$



Fact: if $c > 0$, then for any p,

$$\boxed{\lim_{x \to \infty} x^p e^{-cx} = 0}$$

e.g. even $x^{1000000} e^{-x/7000} \xrightarrow[x \to \infty]{} 0$

(general rule of thumb:

"e^x beats x^p beats $\ln x$"

in terms of behaviour at ∞)