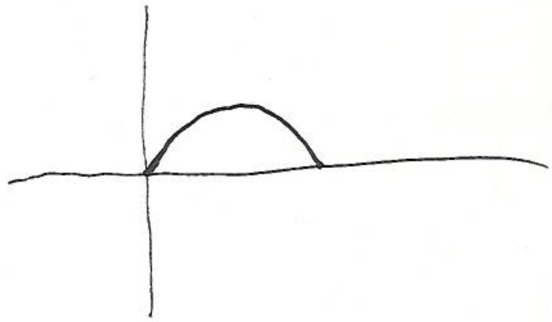
Chain rule

<u>Example</u>: We fire a cannon ball from the origin



at time t after firing, it is at position $(5t, 5t-5t^2)$ $\qquad (0 \leq t \leq 1)$

Let $s(t)$ be the distance of the ball from the origin at time t

so $s(t) = r(x(t), y(t))$

$\quad = r(5t, 5t-5t^2)$ $\quad$ where $r(x,y) = \sqrt{x^2+y^2}$

<u>Qu^n</u>: what is $s'(t) = \frac{d}{dt} s(t)$?

We know $\quad r_x = \frac{x}{\sqrt{x^2+y^2}}$, $\quad r_y = \frac{y}{\sqrt{x^2+y^2}}$

$x'(t) = 5$

$y'(t) = 5 - 10t$

Can we use these to find s' ?

<u>Idea</u>: As t increases, $s(t)$ is ~~increasing~~ changing for "two reasons"

- because $x(t)$ is ~~increasing~~ changing
- because $y(t)$ is changing

Treat these separately:

the rate of change of $s(t)$ with t "due to x"

is $r_x(x(t), y(t))\,x'(t)$

similarly rate of change of $s(t)$ with t "due to y"

is $r_y(x(t), y(t))\,y'(t)$

so total rate = sum of the two

$$= r_x x' + r_y y' = \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt}$$

<u>Generally:</u>

<u>chain rule:</u>

If z is a function of x and y each of which is a function of t, then considering z as a function of t,

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

i.e. given functions $f(x,y)$, $g(t)$, $h(t)$

$$\frac{d}{dt} f(g(t), h(t)) = f_x g' + f_y h'$$

$$= f_x(g(t), h(t))\,g'(t) + f_y(g(t), h(t))\,h'(t)$$

Finishing the example:

$$s'(t) = r_x x' + r_y y'$$

$$= \frac{x}{\sqrt{x^2+y^2}} \cdot 5 + \frac{y}{\sqrt{x^2+y^2}} (5-10t)$$

$$= \frac{25t}{s(t)} + \frac{5t-5t^2}{s(t)} (5-10t)$$

$$= \frac{1}{s(t)} \left(25t + (5t-5t^2)(5-10t) \right)$$

$$f(x,y) = xy$$

$$g(t) = t^2$$

$$h(t) = t^3 - t$$

$$\frac{d}{dt} f(g(t), h(t)) = f_x \cdot g' + f_y \cdot h'$$

$$= y \cdot 2t + x(3t^2-1)$$

$$= h(t)\, 2t + g(t)(3t^2-1)$$

$$= (t^3-t)(2t) + t^2(3t^2-1)$$

$$= 5t^4 - 3t^2$$

(note : $f(g(t), h(t)) = (t^2)(t^3-t)$

$= t^5 - t^3$

$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(t^5 - t^3) = 5t^4 - 3t^2$)

<u>Note</u> : $\frac{d}{dt} f(g(t), h(t)) = 0 \iff t=0$ or $t = \pm\sqrt{3/5}$