Definition: $f(x,y)$ has a local minimum (or relative minimum) at (x_0, y_0)

if for some disc centred at (x_0, y_0)

for every (x,y) in the disc,

$$f(x,y) \geq f(x_0, y_0)$$

(sim for local max (a same def^n, but $\leq$ rather than $\geq$.)

Critical points:

If $f(x,y)$ ~~local m~~ has a local min/max at (x_0, y_0)

then so do the functions of one variable

$$f(x, y_0) \quad \text{and} \quad f(x_0, y)$$

so if $f(x, y_0)$ and $f(x_0, y)$ are differentiable,

then ~~$f'(x)$~~ $\frac{d}{dx} f(x, y_0)$ is 0 at $x = x_0$

and $\frac{d}{dy} f(x_0, y)$ is 0 at $y = y_0$

in other words, $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$

(if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist)

Def^n: $f(x,y)$ is differentiable at point (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are defined

Def^n (x_0, y_0) is a critical point of $f(x,y)$

if f is differentiable at (x_0, y_0)
and $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$

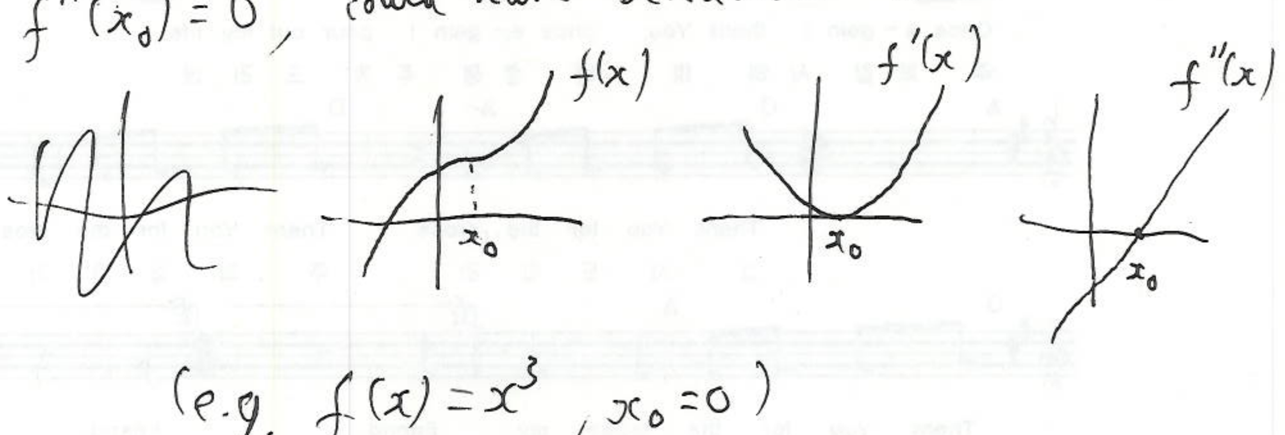
so we've argued:
If (x_0, y_0) is a local min/max of f
and if f is diff'ble at (x_0, y_0)
then (x_0, y_0) is a critical point of f

$\boxed{\text{"min/max} \Rightarrow \text{critical"}}$

Let's ask: when is a critical point a local min/max?

<u>Recall</u>: A (twice differentiable) function of one variable $f(x)$
with a critical point at x_0 (i.e. $f'(x_0) = 0$)
has local min/max at x_0 ~~[illegible] if $f''(x_0) \neq 0$~~
if $f''(x_0) \neq 0$

(if $f''(x_0) = 0$, could have behaviour like



(e.g. $f(x) = x^3$, $x_0 = 0$)

In two variables we have a new phenomenon:

Saddle points

If (x_0, y_0) is a critical ~~pt~~ point of $f(x,y)$

and if $x(t)$, $y(t)$ are differentiable functions

and $x(t_0) = x_0$ ~~&~~ $y(t_0) = y_0$

(so $(x(t), y(t))$ defines a curve in the xy-plane passing ~~(x_0, y_0)~~ through (x_0, y_0))

then the composition $f(x(t), y(t))$ has a critical point at t_0

Indeed: $\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$

is 0 at $t = t_0$

since $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$

If (x_0, y_0) is a local min of f

then t_0 is a local min of $f(x(t), y(t))$

~~Sim for max~~

So "on any curve through (x_0, y_0)" f has a local min at (x_0, y_0)

sim for max

But it can happen that on ~~for~~ some curves through (x_0, y_0), f has a local min,

but on some curves it has a local max

In this case, (x_0, y_0) is a saddle point of f

e.g. $x^2 - y^2$

or $f(x,y) = 3(x+y)^2 - (x-y)^2$

$f_x(x,y) = 6(x+y) - 2(x-y)$ $\qquad f_{xx}(x,y) = 6 - 2 = 4$

$f_y(x,y) = 6(x+y) + 2(x-y)$ $\qquad f_{yy}(x,y) = 6 - 2 = 4$

$f_{xy}(x,y) = 6 - (-2) = 8$

critical points: (x,y) is critical if $f_x(x,y) = 0$ and $f_y(x,y) = 0$

so then $0 = f_x(x,y) + f_y(x,y) = 12(x+y)$

so $x = -y$

and $0 = f_y(x,y) - f_x(x,y) = 4(x-y)$

so $x = y$

so $x = 0$ and $y = 0$

so only critical point is $(0,0)$

if $x(t) = t$, $y(t) = 0$

then $f(x(t), y(t)) = 3t^2 - t^2 = 2t^2$ which has a local min at $t = 0$

i.e. $(0,0)$

but if $x(t) = t$, $y(t) = -t$

then $f(x(t), y(t)) = 3(t-t)^2 - (t-(-t))^2$

$= 0 - (2t)^2$

$= -4t^2$ so local <u>max</u> at $t = 0$

Classifying critical points in terms of 2^{nd} derivatives

- 1-varble - look at sign of f''
- 2-varbles; we've seen it isn't enough to look at signs of f_{xx} and f_{yy}

but we do get information by considering f_{xy} as well:

Second partials test:

If (x_0, y_0) is a critical point of $f(x,y)$

let $D := f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$

If $D<0$, then (x_0,y_0) is a saddle point

If $D>0$, then (x_0,y_0) is a local min or a local max

and then if $f_{xx}(x_0,y_0)>0$ then (x_0,y_0) is a local min

and if $f_{xx}(x_0,y_0)<0$ " " " " max

if $f_{xx}(x_0,y_0)=0$, could be either

If $D=0$, we get no information

Examples: Find and classify the critical points of the following functions

- $f(x,y) = x^2+y^2$

$f_x = 2x \quad f_y = 2y \quad f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0 = f_{yx}$

critical points:

solve $f_x(x,y) = 0 = f_y(x,y)$

$$\begin{aligned} 2x &= 0 \\ 2y &= 0 \end{aligned} \iff x=0,\ y=0$$

so only critical point is $(0,0)$

$D = 2 \cdot 2 - 0^2 = 4 > 0$ so local min/max

$f_{xx}(x_0, y_0) = 2 > 0$ so local min

$f(x,y) = x^2 - y^2$

$f_x = 2x \quad f_y = -2y \quad f_{xx} = 2 \quad f_{yy} = -2 \quad f_{xy} = 0$

critical only at $(0,0)$

$D = 2(-2) - 0 = -4 < 0$ so saddle point