critical points:

solve $f_x(x,y) = 0 = f_y(x,y)$

$$2x = 0 \quad \Longleftrightarrow \quad x=0,\ y=0$$
$$2y = 0$$

so only critical point is $(0,0)$

$D = 2 \cdot 2 - 0^2 = 4 > 0$ so local min/max

$f_{xx}(x_0, y_0) = 2 > 0$ so local min

$f(x,y) = x^2 - y^2$

$f_x = 2x \qquad f_y = -2y \qquad f_{xx} = 2 \quad f_{yy} = -2 \quad f_{xy} = 0$

critical only at $(0,0)$

$D = 2(-2) - 0 = -4 < 0$ so saddle point

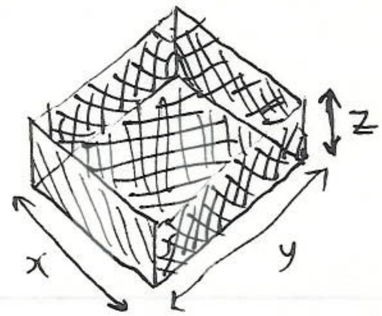
Example: We are designing a cuboid tank, It is to have the top face open, and the front face is to be made of glass and the other four faces are to be made of metal.

The glass weighs three times as much per unit area as the metal

The volume is to be $1 m^3$

We want the tank to be as light as possible.

What should be its dimensions?



Solution: Weight $= xy + 2yz + xz + 3xz$ want to minimize this

Volume $= xyz = 1$

so $z = \frac{1}{xy}$

$$W(x,y) = xy + 2\frac{1}{x} + \frac{1}{y} + 3\frac{1}{y}$$
$$= xy + 2/x + 4/y$$

want to find positive (x,y) at which this is minimal

Find critical values:

$W_x = 0 = W_y$

$W_x(x,y) = y - 2/x^2$ $W_y(x,y) = x - 4/y^2$

solve $\begin{cases} y - 2/x^2 = 0 & (1) \\ x - 4/y^2 = 0 & (2) \end{cases}$

(1) $\Leftrightarrow$ $y = 2/x^2$

(2) $\Leftrightarrow$ $x = 4/y^2$

$$\Rightarrow x = 4/(2/x^2)^2 = 4/(4/x^4) = \frac{4x^4}{4} = x^4$$

and $y = 2/x^2 = 2/(4/y^2)^2 = \frac{2y^4}{16} = \frac{y^4}{8}$

$x = x^4 \iff x = 0$ or $x^3 = 1$

$\iff x = 0$ or $x = 1$

we're looking for positive x, so $x = 1$

so $y = 2/1^2 = 2$

so $(1,2)$ is the only critical point

Let's confirm that it's a local minimum:

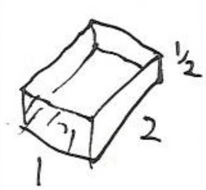
$W_{xx} = 4/x^3 \qquad W_{yy} = 8/y^3 \qquad W_{xy} = 1$

$$\begin{aligned} D(1,2) &= W_{xx}(1,2)\, W_{yy}(1,2) - W_{xy}(1,2)^2 \\ &= \left(4/1^3\right)\left(8/2^3\right) - 1^2 \\ &= 3 > 0 \implies \text{local min/max} \end{aligned}$$

$W_{xx}(1,2) = 4 > 0 \implies$ local min

$(x,y) = (1,2) \implies z = \frac{1}{xy} = \frac{1}{2}$

So the tank should have dimensions



Example: Find and classify the critical points of $f(x,y) = xye^{-(x^2+y^2)}$

$$\begin{aligned} f_x &= ye^{-(x^2+y^2)} + xy(-2x)e^{-(x^2+y^2)} \\ &= (1-2x^2)ye^{-(x^2+y^2)} \end{aligned}$$

$f_y = (1-2y^2)xe^{-(x^2+y^2)}$

(x,y) critical $\Leftrightarrow f_x(x,y) = 0 = f_y(x,y)$

$$f_x(x,y) = 0 \Leftrightarrow (1-2x^2)y = 0$$
$$\Leftrightarrow 1-2x^2 = 0 \text{ or } y = 0$$
$$\Leftrightarrow x = \pm\sqrt{1/2} \text{ or } y = 0$$

$$f_y(x,y) = 0 \Leftrightarrow y = \pm\sqrt{1/2} \text{ or } x = 0$$

so our critical points are:

$(0,0)$, $(-\sqrt{1/2}, \sqrt{1/2})$, $(-\sqrt{1/2}, -\sqrt{1/2})$, $(\sqrt{1/2}, \sqrt{1/2})$ $(\sqrt{1/2}, -\sqrt{1/2})$

$f_{xx} = -4xye^{-(x^2+y^2)} - 2x(1-2x^2)ye^{-(x^2+y^2)}$
$\phantom{f_{xx}} = (4x^2-6x)ye^{-(x^2+y^2)}$

$f_{yy} = (4y^2-6y)xe^{-(x^2+y^2)}$

$f_{xy} = (1-2y^2)(1-2x^2)e^{-(x^2+y^2)}$

$(0,0)$; $D(0,0) = f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}(0,0)^2$
$= 0 \quad 0 \quad - 1^2$
$= -1$ - so saddle point

$(\sqrt{1/2}, \sqrt{1/2})$; $D(\sqrt{1/2}, \sqrt{1/2}) = ((2-6\sqrt{1/2})\sqrt{1/2}\, e^{-(1)})^2 - 0$

> 0 min/max

$f_{xx}: 2\sqrt{1/2} - 3 < 0$ max

General Problem:

Given a function $f(x,y)$

and a constraint $g(x,y)=k$,

find the greatest value of $f(x,y)$

as (x,y) varies amongst those values such that the constraint $g(x,y)=k$ is satisfied,

and find (x,y) where this constrained maximum is attained

(sim for minima)

Remarks: · Constrained maxima may well not be maxima of $f(x,y)$ in the usual (unconstrained) sense, so our critical point methods don't help

· Sometimes we can reduce the problem to a 1-variable optimisation problem, e.g. by solving for y in $g(x,y)=k$

(e.g. Example 1, $g(x,y)=x+y$, $k=10$, so $y=10-x$)

and then optimise

$$f(x, y(x))$$

But often this is not straightforward.

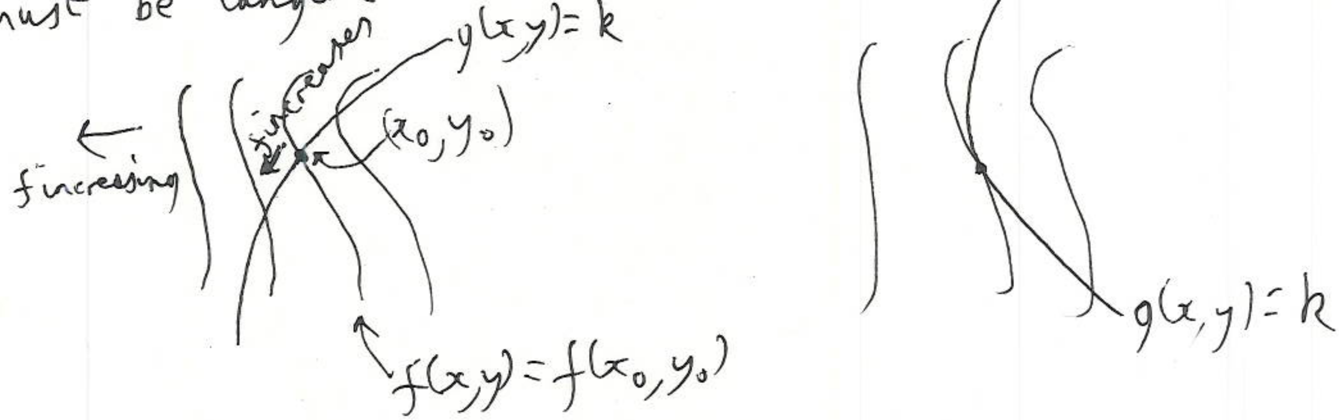
Lagrangian Multipliers

Suppose (x_0, y_0) is a constrained maximum of $f(x,y)$ subject to a constraint curve $g(x,y)=k$

Then the level curve of f passing through (x_0,y_0)

$$\left(f(x,y) = f(x_0,y_0) \right)$$

must be tangent to the constraint curve $g(x,y)=k$



Now note: given a point (x_0,y_0) on a curve defined by

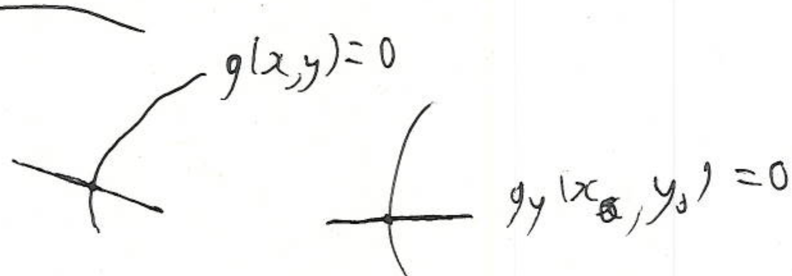
$$g(x,y)=k$$

if $\left(g_x(x_0,y_0),\ g_y(x_0,y_0)\right) \neq (0,0)$

then the straight line passing through

$(0,0)$ and $\left(g_x(x_0,y_0),\ g_y(x_0,y_0)\right)$

is perpendicular to the curve at (x_0,y_0)



So since the level curve $f(x,y) = f(x_0, y_0)$

and the constraint curve $g(x,y) = k$

are tangent, these lines given by

the first derivatives are the same.

So for some λ, ← "Lagrangian multiplier"

$$f_x(x_0, y_0) = \lambda g_x(x_0, y_0)$$
$$f_y(x_0, y_0) = \lambda g_y(x_0, y_0)$$

("$(f_x, f_y) = \lambda (g_x, g_y)$ at (x_0, y_0)")

So to solve the constrained optimisation problem: maximise $f(x,y)$ subject to the constraint $g(x,y) = k$;

we can proceed as follows:

- Solve the three equations in three unknowns:

$$f_x(x,y) = \lambda g_x(x,y)$$
$$f_y(x,y) = \lambda g_y(x,y)$$
$$g(x,y) = k$$

- Generally we get finitely many solutions
- Then (assuming, as we generally can, that there is a constrained maximum and that $(g_x, g_y) \neq (0,0)$ there), the maximum is at one of the solutions to the system of 3 equations.
- So calculate $f(x,y)$ at each solution, and see which is greatest

~~Consider~~

Constrained optimisation

Example 1:

You suddenly remember that you have an important test tomorrow ~~test~~ morning.

You have 10 hours in which to revise and sleep.

You estimate that if you revise for r hours and sleep for s hours,

your mark to be $10 s^{1/5} r^{4/5}$ percent

How long should spend on each activity

Solution: We have 10 hours to divide between revising and sleeping

i.e $r + s \leq 10$

clearly to get the best mark we should spend all 10 hours either revising or sleeping, so

$$r + s = 10$$

So we want to solve the constrained optimisation problem:

maximise $f(r,s) = 10 s^{1/5} r^{4/5}$

with constraint $g(r,s) = r + s = 10$ "k"

Solution method 1: reduce to 1 variable:

$$r + s = 10 \iff s = 10 - r$$

so we want to maximise $f(r, 10-r) = 10(10-r)^{1/5} r^{4/5}$

so: solve $\frac{d}{dr} f(r, 10-r) = 0 \ldots$

(Note: the explanation I just gave for why this works is not on the syllabus, the technique itself is)

Solution method 2: Lagrangian multipliers:

At the max (r_0, s_0), we know that the derivatives of f and g are proportional, i.e. for some λ

$$f_r(r_0, s_0) = \lambda g_r(r_0, s_0) \quad \text{i.e.} \quad \frac{4}{5} 10 s_0^{1/5} r_0^{-1/5} = \lambda$$

$$f_s(r_0, s_0) = \lambda g_s(r_0, s_0) \quad \frac{1}{5} 10 s_0^{-4/5} r_0^{4/5} = \lambda$$

also, $r_0 + s_0 = 10$

so $\frac{4}{5} 10 s_0^{1/5} r_0^{-1/5} = \frac{1}{5} 10 s_0^{-4/5} r_0^{4/5}$

so $4 s_0^{1/5} r_0^{-1/5} = s_0^{-4/5} r_0^{4/5}$

so $4 s_0 = r_0$

but $r_0 = 10 - s_0$

so $4 s_0 = 10 - s_0$

so $s_0 = 2$

so $r_0 = 8$

$s_0 = 2$, $r_0 = 8$ is the only solution to our system of equations, so must be the max.

So answer: sleep for 2 hours, revise for the other 8

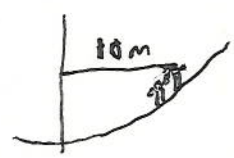
Example 2: A goat is leashed with a 10m long rope to a pole in a valley

The height of the ground x metres East and y metres North of the pole is

$$h(x,y) = \frac{x^2 + 2y^2}{100} + \frac{x}{10} + 1$$

How high can the goat get without breaking the rope

Solution: The goat has to stay within 10m of the pole, horizontally



So if the goat stands at co-ordinates (x,y), we have $\sqrt{x^2+y^2} \leq 10$

i.e. $x^2+y^2 \leq 100$

Since the goat is in a valley, clearly the highest point the goat can get to will be on the perimeter of this circle,

i.e. $x^2+y^2=100$

So we want to find the max of

$$h(x,y) = \frac{x^2+2y^2}{100} + x/10 + 1$$

subject to the constraint $g(x,y) = x^2+y^2 = 100$

so let's solve

$$\begin{cases} h_x(x,y) = \lambda\, g_x(x,y) \\ h_y(x,y) = \lambda\, g_y(x,y) \\ g(x,y) = 100 \end{cases} \quad \text{i.e.} \begin{cases} \frac{2x}{100} + \frac{1}{10} = \lambda 2x & ① \\ \frac{4y}{100} = \lambda 2y & ② \\ x^2 + y^2 = 100 & ③ \end{cases}$$

② $\Rightarrow$ if $y \neq 0$ then $\lambda = \frac{4y}{(100)2y} = \frac{1}{50}$

so if $y \neq 0$, ① $\Leftrightarrow \frac{2x}{100} + \frac{1}{10} = \frac{2x}{50}$

$\Leftrightarrow 2x\left(\frac{1}{50} - \frac{1}{100}\right) = \frac{1}{10}$

$\Leftrightarrow 2x = 10$

$\Leftrightarrow x = 5$

so ③ $\Leftrightarrow y^2 = 100 - 25 = 75$

$\Leftrightarrow y = \pm\sqrt{75}$

If $y = 0$; ② holds,

③ $\Leftrightarrow x^2 = 100 \Leftrightarrow x = \pm 10$

and ① holds for some λ

So we have 4 solutions : $(5, \pm\sqrt{75})$ and $(\pm 10, 0)$

so max height occurs at one of these points

$h(5, \pm\sqrt{75}) = \frac{5^2 + 2(75)}{100} + \frac{25}{10} + 1 = 5.25$

$h(\pm 10, 0) = 3$

$5.25 > 3$ so 5.25 is the max height

Differential Equations

A differential equation is something like

$$\frac{dy}{dx} = x \quad \text{or} \quad \frac{dy}{dx} = y \quad \text{or} \quad y^2 \frac{dy}{dx} + x = 0$$

$$\text{or} \quad \frac{dy}{dx}\left(y + e^{\left(x \frac{dy}{dx}\right)}\right) = xy$$

- some equation expessing some algebraic relation between a variable x, a function y of x and its derivatives with respect to x

Solving a differential equation means finding y as a function of x such that the equation holds

e.g. given $\frac{dy}{dx} = 1$

$y = x$ is a solution

so is $y = x + 2$

These solutions are called "<u>particular solutions</u>"

The <u>general solution</u> is one like "$y = x + c$", describing completely the whole family of solutions

We've already spent a lot of time thinking a certain kind of differential equation:

$$\frac{dy}{dx} = f(x)$$

- the solutions are precisely the anti-derivatives and the general solution is $F(x)+c$ where $F(x)$ is any particular solution

<u>Terminology</u>: A differential equation along with a condition of the form "$y=a$ when $x=b$" is an <u>initial value problem</u> - expect to only get one solution

e.g. $\frac{dy}{dt}=t$, $y=1$ when $t=0$

<u>solution</u>: general solution to $\frac{dy}{dt}=t$ is $\frac{t^2}{2}+c$

but $\frac{0^2}{2}+c=1$

so $c=1$

so $\frac{t^2}{2}+1$ is the only solution

<u>Separable Differential Equations</u>:

We can solve a differential equation of the form

$$f(y)\frac{dy}{dx}=g(x)$$

by solving $\int f(y)\,dy = \int g(x)\,dx$

e.g. ~~[illegible]~~

e.g. $y\frac{dy}{dx} = e^x$

$$\Leftrightarrow \int y\,dy = \int e^x\,dx$$

$$\Leftrightarrow \frac{y^2}{2} = e^x + c$$

$$\Leftrightarrow y^2 = 2e^x + 2c$$

$$\Leftrightarrow y = \sqrt{2e^x + 2c}$$

check: $\frac{dy}{dx} = \frac{\frac{1}{2}2e^x}{\sqrt{2e^x+2c}}$

$= e^x / \sqrt{2e^x+2c}$

so $y\frac{dy}{dx} = \sqrt{2e^x+2c} \cdot \frac{e^x}{\sqrt{2e^x+2c}}$

$= e^x$ ✓

(why this works in general:

if $\int f(y)\,dy = z = \int g(x)\,dx$

then $g(x) = \underbrace{\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}}_{\text{chain rule}} = f(y)\frac{dy}{dx}$.)

Given a nasty-looking differential equation, we can sometimes manipulate to one of the form $f(y)\frac{dy}{dx} = g(x)$. These are called <u>separable</u> differential equations

<u>Example</u>: $\frac{dy}{dx} = y$: $\frac{dy}{dx} = y \Leftrightarrow \frac{1}{y}\frac{dy}{dx} = 1$

$$\Leftrightarrow \int \frac{1}{y}\,dy = \int 1\,dx$$

$$\Leftrightarrow \log y = x + c \Leftrightarrow y = e^{x+c}$$

Note $e^{x+c} = e^x e^c$

so we could also write the general solution

as $y = Ae^x$

- $\frac{dy}{dx} = xe^{x-y}$:

$$\frac{dy}{dx} = xe^{x-y} = x\frac{e^x}{e^y}$$

$$\Leftrightarrow \quad e^y \frac{dy}{dx} = xe^x$$

$$\Leftrightarrow \quad \int e^y dy = \int xe^x dx$$

$$\Leftrightarrow \quad e^y = (x-1)e^x + c$$

$$\Leftrightarrow \quad y = \log\left((x-1)e^x + c\right)$$

$$\int xe^x dx = xe^x - \int e^x dx = (x-1)e^x$$