Fact: Limits are unique if they exist

i.e. if $\lim_{x \to b} f(x) = c_1$ and $\lim_{x \to b} f(x) = c_2$

then $c_1 = c_2$

Notation:

$f(x) \to c$ as $x \to b$

$f(x) \xrightarrow[x \to b]{} c$

$\lim_{x \to b} f(x) = c$

The limit of $f(x)$ as x tends to b

mean the same thing

"$f(x)$ has no finite limit as x tends to b"

means that it is not true for any c

then $f(x) \xrightarrow[x \to b]{} c$

Variations

One-sided limits:

$\lim_{x \to b^+} f(x) = c$ means that $f(x)$ is arbitrarily close to c for all x greater than b and sufficiently close to b

$\lim_{x \to b^-} f(x) = c$

similar but only look at $x < b$

Limits at infinity:

$\lim_{x \to +\infty} f(x) = c$ means that $f(x)$ is arbitrarily close to c for all sufficiently positive x

$\cancel{\lim_{x \to +\infty} (1+e^{-x})}$

$$\lim_{x \to +\infty} e^{-x} = 0$$

$$\lim_{x \to -\infty} e^{-x} = \infty$$

Limit laws — see section 2.3

things like: $\lim_{x \to b} f(x)g(x) = \left(\lim_{x \to b} f(x)\right)\left(\lim_{x \to b} g(x)\right)$

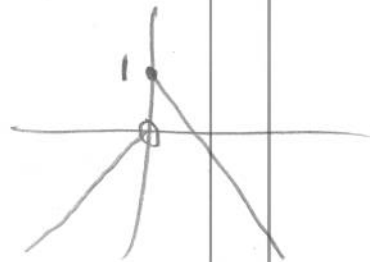
Continuity

Definition: A function f is continuous at b if $\lim_{x \to b} f(x)$ exists and f is defined at b

and $\boxed{\lim_{x \to b} f(x) = f(b)}$

Example:

$$f(x) = \begin{cases} x & x < 0 \\ 1-x & x \geq 0 \end{cases}$$



$\lim\limits_{x \to 0} f(x)$ does not exist

but $\lim\limits_{x \to 0^+} f(x) = 1$ $\qquad \lim\limits_{x \to 0^-} f(x) = 0$

Infinite limits

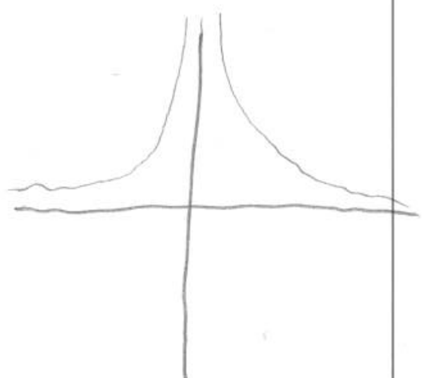
$\lim\limits_{x \to b} f(x) = +\infty$ means that $f(x)$ is arbitiarily positive for all x sufficiently close to but not equal to b

" $= -\infty$ " " " " negative " " "

" " " " " " "

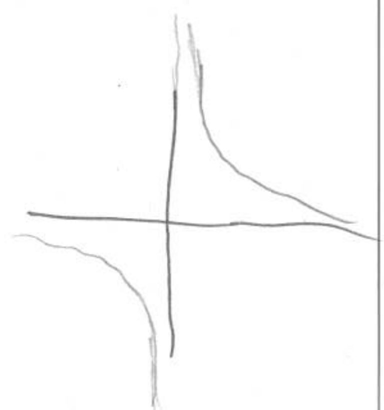
Examples:

$\lim\limits_{x \to 0} x^{-2} = +\infty$



$\lim\limits_{x \to 0^+} x^{-1} = +\infty$

$\lim\limits_{x \to 0^-} x^{-1} = -\infty$

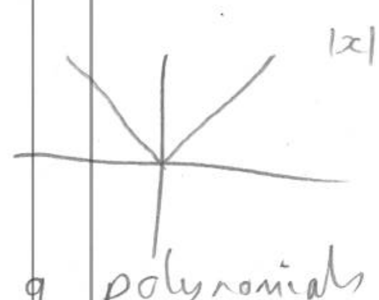


Examples:

The following are all continuous at every point in their domains:

domain $\mathbb{R}$

- polynomials x, x^3, $x^5 - 7x + 37$
- exp $x \mapsto e^x$
- sin, cos, arctan
- $\text{abs}(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



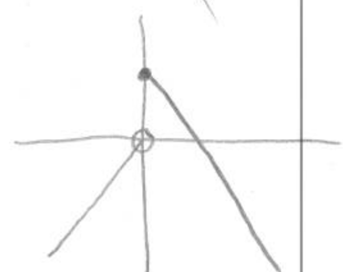
- rational functions ($\frac{p(x)}{q(x)}$ p, q polynomials)

  e.g. $\frac{x^2}{3+x}$ domain: $\mathbb{R}$ except -3

  not defined at -3 so not continuous at -3

- tan, log, arcsin, arccos

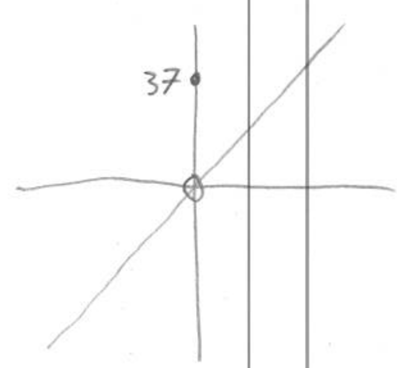
- $f(x) := \begin{cases} x & x < 0 \\ 1-x & x \geq 0 \end{cases}$



not continuous at 0

since has no finite limit at 0

- $f(x) := \begin{cases} x & x \neq 0 \\ 37 & x = 0 \end{cases}$



$\lim_{x \to 0} f(x) = 0$

but $f(0) = 37$

so not continuous at 0

• $f(x) = x \sin \frac{1}{x}$ not continuous at 0 since not defined

but if we define

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

then $g(x)$ is continuous on $\mathbb{R}$

Fact: If $\lim_{x \to b} g(x)$ exists and if f is continuous at $\lim_{x \to b} g(x)$

$$\lim_{x \to b} f(g(x)) = f\left(\lim_{x \to b} g(x)\right)$$

Example:

$$\lim_{x \to 0} e^{x \sin \frac{1}{x}}$$

$$= e^{\lim_{x \to 0} x \sin \frac{1}{x}}$$

$$= e^{0}$$

$$= 1$$

$f(x) = e^{x}$

$g(x) = x \sin \frac{1}{x}$

$\lim_{x \to 0} g(x) = 0$

$f(x)$ is cont^{s} at 0