

(first lecture of wk10 is contained in wk9.pdf)  
 (second lecture, covering volumes of revolution and averages, was taught for me by Dr. Haskell - see scans2.pdf)

## Work

**Warning:** I largely follow Stewart here. This is very much a “Just-So” account of the physics involved. I will leave it to your physics lecturers to give you the full story.

**Definition:** Suppose a particle moves along a line from  $x = a$  to  $x = b$  while a force acts on the particle along the line of motion, and suppose the force in the direction of motion is a continuous function  $F(x)$  of position. Then the *work done* by the force is

$$W = \int_a^b F(x) dx.$$

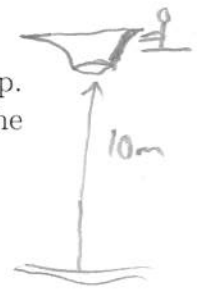


If  $x$  and  $F$  are measured in metres and Newtons respectively, then  $W$  is measured in Joules.

**Example:** I hold a 5000kg anvil 10m above the ground, then let it drop. As it falls, it is acted on by gravity with a force of  $g = 9.8$  Newtons. So the work done by gravity as the anvil falls is

$$\int_{10}^0 -mg dh = \int_{10}^0 -490000 dh = 4900000$$

Joules, i.e. ~~49MJ~~ **0.49MJ**  
 (Enough to power a 15W ~~light bulb~~ **LED** for nearly 4 days)



**Generally:** for a **constant** force in the direction of motion, work is force times distance travelled

$$W = Fd;$$

integration is needed when the force is varying.

**Example:** How much work do I have to do to reset the previous experiment - i.e. lift the anvil back up 10m?

In lifting it, I have to counteract the force of gravity - but can apply arbitrarily small forces beyond that. So the work I have to do is

$$\int_0^{10} mg dh = \int_0^{10} 490000 dh = 4900000.$$

(Fact: it actually doesn't matter what forces I apply in lifting the anvil; as long as it ends up at rest 10m above the ground, the work done by the forces will be the same, 4.9MJ.)

**Example:** The force exerted by a compressed or stretched spring is approximately proportional to the difference between its length and its relaxed length

$$F = -kx$$

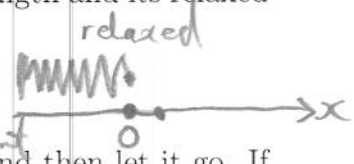
(Hooke's law).

I pull back the spring on a pinball table by 0.05m and then let it go. If  $k = 1000$  for this spring, what is the work done by the spring as it springs back?

(Again, it turns out that this is the same as the work I had to do in pulling back the spring.)

Solution:

$$\begin{aligned} \int_{-0.05}^0 -kx dx &= \int_{-0.05}^0 -1000x dx \\ &= -1000 \left[ \frac{x^2}{2} \right]_{-0.05}^0 \\ &= -1000 * (0 - 0.0025/2) = 1.25 J \end{aligned}$$



**Example:** The CN tower collapses. How much energy is released by the falling tower?

Solution: for simplicity, let's assume that the entirety of the tower falls to ground level.

We take the height of the tower to be  $H = 500$  metres and its total mass to be  $120 * 10^6$  kg.

First, suppose we know that how the mass of the tower is distributed along its height. Suppose that at height  $h$ , the mass per unit height is  $M(h)$   $kg m^{-1}$ .

Then the energy is an integral: if we divide the tower into  $n$  segments of equal length  $\Delta_n = \frac{H}{n}$ , and estimate the mass of the  $i^{th}$  as  $M(h_i^*) \Delta_n$ , then as in the anvil example we estimate the energy released by the falling of the  $i^{th}$  segment as  $M(h_i^*) \Delta_n g h_i^*$ , and hence the entire energy as

$$\sum_{i=1}^n M(h_i^*) \Delta_n g h_i^*$$

*mass of segment*



Taking the limit as  $n \rightarrow \infty$ , we get

$$g \int_0^H h M(h) dh.$$

Now let's estimate  $M(h)$ : for simplicity, let's assume a linear relationship. (So in particular we entirely ignore the SkyPod...) So if the mass per unit height at height  $h$  is  $M(h) = (500 - h)k$ , then

$$120 * 10^6 = \int_0^{500} (500 - h) k dh = k \left[ 500h - \frac{h^2}{2} \right]_0^{500} = k \frac{500^2}{2}$$

So

$$k = \frac{120 * 10^6}{\frac{500^2}{2}} = 960 kg m^{-1}.$$

So the work done by gravity as the tower falls is

$$\begin{aligned}
 g \int_0^{500} h(500-h) k dh &= kg \left[ \frac{500h^2}{2} - \frac{h^3}{3} \right]_0^{500} \\
 &= kg \frac{500^3}{6} \\
 &= 960 * 9.8 * \frac{500^3}{6} \\
 &= 0.39 * 10^{12} J \\
 &= 0.39 TJ
 \end{aligned}$$

(Again, this is also the energy spent fighting gravity to construct the tower... for comparison, the Hiroshima bomb is said to have released 67TJ.)

## Integration by parts

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

so

$$\int (f'(x)g(x) + f(x)g'(x)) dx = f(x)g(x) + C$$

so

$$\int f'(x)g(x) dx = f(x)g(x) - \int f(x)g'(x) dx + C$$

**Example:**

$$\begin{aligned}
 &f(x) = \sin(x), \quad g(x) = x \\
 \int x \cos(x) dx &= \int \sin'(x) x dx = \sin(x)x - \int \sin(x) 1 dx = x \sin(x) + \cos(x) + C.
 \end{aligned}$$

**Remark:** Setting  $u = f(x)$  and  $v = g(x)$  (and assuming  $u$  is a function of  $v$  and  $v$  is a function of  $u$ ) we can use the Substitution rule to rewrite this as

$$\int v du = uv - \int u dv.$$