(first lecture of wk10 is contained in wk9.pdf)

(second lecture, covering volumes of revolution and averages, was taught for me by Dr. Haskell - see scans2.pdf)

Work

Warning: I largely follow Stewart here. This is very much a "Just-So" account of the physics involved. I will leave it to your physics lecturers to give you the full story.

Definition: Suppose a particle moves along a line from $x = a$ to $x = b$ while a force acts on the particle along the line of motion, and suppose the force in the direction of motion is a continuous function $F(x)$ of position. Then the *work done* by the force is

$$W = \int_a^b F(x)\, dx.$$

If x and F are measured in metres and Newtons respectively, then W is measured in Joules.

Example: I hold a 5000kg anvil 10m above the ground, then let it drop. As it falls, it is acted on by gravity with a force of $g = 9.8$ Newtons. So the work done by gravity as the anvil falls is

$$\int_{10}^{0} -mgdh = \int_{10}^{0} -490000dh = 4900000$$

Joules, i.e. 4.9MJ.

(Enough to power a 15W light bulb for nearly 4 days)

Generally: for a **constant** force in the direction of motion, work is force times distance travelled

$$W = Fd;$$

integration is needed when the force is varying.

Example: How much work do I have to do to reset the previous experiment - i.e. lift the anvil back up 10m?

In lifting it, I have to counteract the force of gravity - but can apply arbitrarily small forces beyond that. So the work I have to do is

$$\int_0^{10} mgdh = \int_0^{10} 490000dh = 4900000.$$

(Fact: it actually doesn't matter what forces I apply in lifting the anvil; as long as it ends up at rest 10m above the ground, the work done by the forces will be the same, 4.9MJ.)

Example: The force exerted by a compressed or stretched spring is approximately proportional to the difference between its length and its relaxed length

$$F = -kx$$

(Hooke's law).

I pull back the spring on a pinball table by 0.05m and then let it go. If $k = 1000$ for this spring, what is the work done by the spring as it springs back?

(Again, it turns out that this is the same as the work I had to do in pulling back the spring.)

Solution:

$$\begin{aligned} \int_{-0.05}^{0} -kxdx &= \int_{-0.05}^{0} -1000xdx \\ &= -1000 \left[\frac{x^2}{2} \right]_{-0.05}^{0} \\ &= -1000 * (0 - 0.0025/2) = 1.25J \end{aligned}$$

Example: The CN tower collapses. How much energy is released by the falling tower?

Solution: for simplicity, let's assume that the entirety of the tower falls to ground level.

We take the height of the tower to be $H = 550$ metres and its total mass to be $120 * 10^6$ kg.

First, suppose we know that how the mass of the tower is distributed along its height. Suppose that at height h, the mass per unit height is $M(h)\, kgm^-1$.

Then the energy is an integral: if we divide the tower into n segments of equal length $\Delta_n = \frac{H}{n}$, and estimate the mass of the i^{th} as $M(h_i^*)\,\Delta_n$, then as in the anvil example we estimate the energy released by the falling of the i^{th} segment as $M(h_i^*)\,\Delta_n g h_i^*$, and hence the entire energy as

$$\sum_{i=1}^{n} M(h_i^*)\,\Delta_n g h_i^*.$$

Taking the limit as $n \to \infty$, we get

$$g \int_0^H hM(h)\, dh.$$

Now let's estimate $M(h)$: for simplicity, let's assume a linear relationship. (So in particular we entirely ignore the SkyPod...) So if the mass per unit height at height h is $M(h) = (500 - h)\, k$, then

$$120 * 10^6 = \int_0^{500} (500 - h)\, kdh = k \left[500h - \frac{h^2}{2} \right]_0^{500} = k \frac{500^2}{2}$$

So

$$k = \frac{120 * 10^6}{\frac{500^2}{2}} = 960kgm^{-1}.$$

So the work done by gravity as the tower falls is

$$\begin{aligned} g\int_0^{500} h\,(500-h)\,k dh &= kg\left[\frac{500h^2}{2}-\frac{h^3}{3}\right]_0^{500} \\ &= kg\frac{500^3}{3} \\ &= 960 * 9.8 * \frac{500^3}{3} \\ &= 0.39 * 10^{12} J \\ &= 0.39 TJ \end{aligned}$$

(Again, this is also the energy spent fighting gravity to construct the tower... for comparison, the Hiroshima bomb is said to have released 67TJ.)

Integration by parts

$$\frac{d}{dx} f(x)\, g(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

so

$$f(x)\, g(x) = \int \left(f'(x)\, g(x) + f(x)\, g'(x)\right) dx$$

so

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx.$$

Example:

$$\int x\cos(x)\, dx = \int \sin'(x)\, x dx = \sin(x)\, x - \int \sin(x)\, 1 dx = x\sin(x) + \cos(x) + C.$$

Remark: Setting $u = f(x)$ and $v = g(x)$ (and assuming u is a function of v and v is a function of u) we can use the Substitution rule to rewrite this as

$$\int v du = uv - \int u dv.$$