Integration by parts

$$\frac{d}{\mathrm{dx}} f(x)\, g(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

so

$$\int \left(f'(x)\, g(x) + f(x)\, g'(x)\right) dx = f(x)\, g(x) + C$$

so

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx.$$

Example:

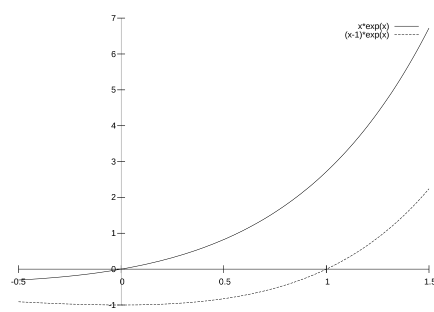
$$\int x \cos(x)\, dx = \int \sin'(x)\, x dx = \sin(x)\, x - \int \sin(x)\, 1 dx = x \sin(x) + \cos(x) + C.$$

Remark: Setting $u = f(x)$ and $v = g(x)$ (and assuming u is a function of v and v is a function of u) we can use the Substitution rule to rewrite this as

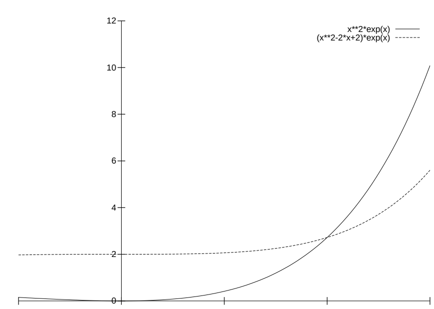
$$\int v\, \mathrm{du} = uv - \int u\, \mathrm{dv}\, .$$

Examples:

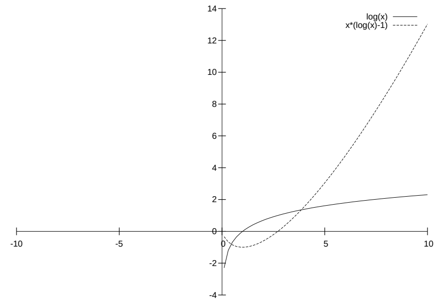
- $\int x e^x dx$



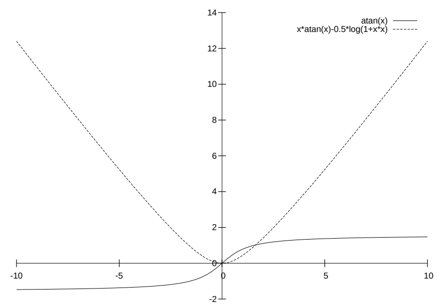
- $\int_0^1 x^2 e^x dx$



- $\int \ln x dx$

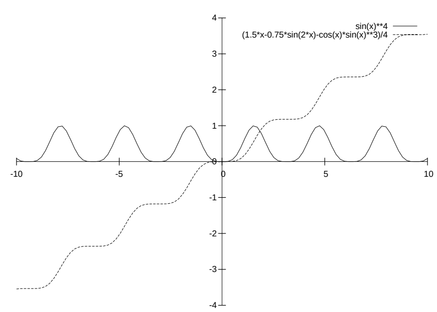


- $\int \arctan x dx$



Trigonometric integrals

- $\int \sin^4(x)\, dx$

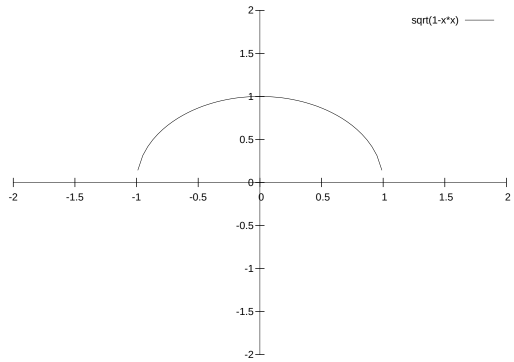


- $\int \sin^2(x) \cos(x)\, dx$
- $\int \sin^3(x)\, dx$
- $\int \sin^4(x) \cos^3(x)\, dx$
- $\int \tan(x) \sec^2(x)\, dx$
- $\int \tan^3(x) \sec^3(x)\, dx$

(Recall: $\sin^2(x) + \cos^2(x) = 1, \sin(2x) = 2\sin(x)\cos(x),\ \cos(2x) = \cos^2(x) - \sin^2(x)$)

Trigonometric substitution

Consider (again) $\int \sqrt{1-x^2}\,\mathrm{dx}$.



Note this only makes sense for x in $[-1,1]$.

Recall Pythagoras.

So if $x = \sin(\theta)$,

$$\int \sqrt{1-x^2}\,\mathrm{dx} = \int \cos(\theta)\,\mathrm{dx}\,.$$

To get an integral involving only θ, we can use the substitution rule backwards. $\frac{\mathrm{dx}}{\mathrm{d}\,\theta} = \cos(\theta)$, so by the substitution rule

$$\begin{aligned}\int \cos(\theta)\cos(\theta)\,\mathrm{d}\,\theta &= \int \sqrt{1-x^2}\frac{\mathrm{dx}}{\mathrm{d}\,\theta}\,\mathrm{d}\,\theta\\ &= \int \sqrt{1-x^2}\,\mathrm{dx}\,.\end{aligned}$$

So

$$\begin{aligned}\int \sqrt{1-x^2}\,\mathrm{dx} &= \int \cos(\theta)\cos(\theta)\,\mathrm{d}\,\theta\\ &= \int \cos^2(\theta)\,\mathrm{d}\,\theta\\ &= \frac{1}{2}\int (1+\cos(2\theta))\,\mathrm{d}\,\theta \qquad = \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C\\ &= \frac{1}{2}(\arcsin(x) + \sin(\theta)\cos(\theta)) + C\\ &= \frac{1}{2}\left(\arcsin(x) + x\sqrt{1-x^2}\right) + C.\end{aligned}$$

So e.g. we recover

$$\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{dx} = \frac{1}{2}\left(\frac{\pi}{2} + 1\sqrt{1-1^2} - \left(-\frac{\pi}{2} + (-1)\sqrt{1-(-1)^2}\right)\right) = \frac{\pi}{2}.$$

$\int \sqrt{c-x^2}\,\mathrm{dx}$ **with** c **positive:** Let $a = \sqrt{c}$, so $\sqrt{c-x^2} = \sqrt{a^2-x^2}$, think about a circle of radius a, and do inverse substitution with $x = a\sin(\theta)$:

$$\int a\cos(\theta)\,a\cos(\theta)\,\mathrm{d}\,\theta = \int \sqrt{a^2-x^2}\,\mathrm{dx}$$

Functions involving $\sqrt{a^2 - x^2}$: e.g.

$$\int x^3\sqrt{a^2-x^2}\,\mathrm{dx} = \int (a\sin(\theta))^3\,(a\cos(\theta))\,(a\cos(\theta))\,\mathrm{d}\,\theta$$

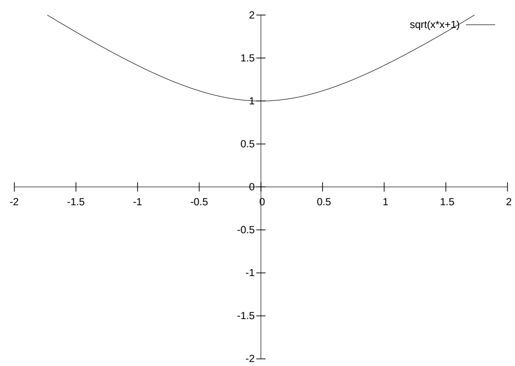
Definite integrals:

$$\int_b^c x^3\sqrt{a^2-x^2}\,\mathrm{dx} = \int_{\arcsin(b/a)}^{\arcsin(c/a)} (a\sin(\theta))^3\,(a\cos(\theta))\,a\cos(\theta)\,\mathrm{d}\,\theta$$

where recall we chose arcsin to have range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

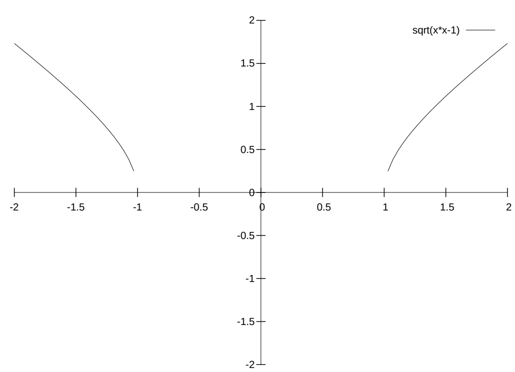
Handling other signs:

- $\sqrt{a^2 + x^2}$:



  try inverse substitution $x = a\tan(\theta)$; so $\sqrt{a^2 + x^2} = a\sec(\theta)$.
  (Using $\tan^2 + 1 = \sec^2$)

- $\sqrt{x^2 - a^2}$:



  try inverse substitution $x = a\sec(\theta)$; so $\sqrt{x^2 - a^2} = a\tan(\theta)$.

- $\sqrt{-x^2 - a^2}$: never defined!

Examples:

- $\int_2^4 x^2\sqrt{x^2-4}\,\mathrm{dx}$
- $\int_2^4 x\sqrt{x^2-4}\,\mathrm{dx}$
- $\int x\left(x^2-2+\pi\right)^{\frac{3}{2}}\,\mathrm{dx}$