Partial fractions

Definition: A rational function is one of the form $\frac{P(x)}{Q(x)}$ where P and Q are polynomials.

e.g. $\frac{x^3-2x+1}{x^2-7}, x^2, \frac{1}{x^9+1}$

We already know how to integrate some of these.

Recall:

- $\int x^n \,\mathrm{dx} = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$
- $\int \frac{1}{x} \,\mathrm{dx} = \ln|x| + C$
- $\int \frac{1}{x^2+1} \,\mathrm{dx} = \arctan x + C$
- $\int \frac{2x}{x^2+1} \,\mathrm{dx} = \int \frac{1}{u} du = \ln|x^2+1| + C$

How about $\int \frac{1}{x^2-1} \,\mathrm{dx}$?

$$\begin{aligned}\int \frac{1}{x^2-1}\,\mathrm{dx} &= \int \frac{1}{(x+1)(x-1)}\,\mathrm{dx} \\ &= \int \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)\mathrm{dx} \\ &= \frac{1}{2}\left(\int \frac{1}{x-1}\,\mathrm{dx} - \int \frac{1}{x+1}\right)\mathrm{dx} \\ &= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) \\ &= \ln\sqrt{\frac{|x-1|}{|x+1|}}\end{aligned}$$

The key trick here was to recognise the rational function $\frac{1}{(x+1)(x-1)}$ as being a linear combination of simpler rational functions ("partial fractions"), namely $\frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$.

We will see that this technique, along with the integrals recalled above and polynomial division, allows us to integrate **any** rational function.

Example of using polynomial division:

$$\begin{aligned}\int \frac{x^3+1}{x^2+1}\,\mathrm{dx} &= \int \frac{x\left(x^2+1\right) - x + 1}{x^2+1} \\ &= \int \left(x + \frac{1-x}{x^2+1}\right)\mathrm{dx} \\ &= \int \left(x + \frac{1}{x^2+1} - \frac{x}{x^2+1}\right)\mathrm{dx} \\ &= \frac{x^2}{2} + \arctan(x) - \frac{\ln|x^2+1|}{2} + C\end{aligned}$$

Fact I: Any polynomial over the reals can be factored as a product of linear and quadratic factors.

Proof: Consequence of the "Fundamental Theorem of Algebra" - ask your 1ZC3 lecturer about it next semester.

Examples: $(-1)^3+1=0$ so $(x+1)$ divides x^3+1

$x^3+1=(x+1)(x^2-x+1)$

- $x^3+1=$ ~~$(x+1)(x^2+x-1) = (x+1)\left(x+\frac{-1+\sqrt{5}}{2}\right)\left(x+\frac{-1-\sqrt{5}}{2}\right)$~~
- $x^4-1=(x^2+1)(x^2-1)=(x^2+1)(x+1)(x-1)$

does (x^2-x+1) factor

$b^2-4ac=(-1)^2-4$

$=-3<0$

so does not factor

Fact II: Given $\frac{P(x)}{Q(x)}$ with $\deg(P)<\deg(Q)$, if $Q(x)$ factors into distinct irreducible factors

$$Q(x)=q_1(x)\,q_2(x)\ldots q_n(x)$$

(each q_i linear or irreducible quadratic) then there exist $p_i(x)$ with $deg(p_i) < deg(q_i)$ such that

$$\frac{P(x)}{Q(x)}=\sum_i \frac{p_i(x)}{q_i(x)}.$$

Proof: Can be proven quite easily with linear algebra. Ask your 1ZC3 lecturer about this too!

Examples:

- $\frac{1}{x^3-x} = \frac{1}{x(x+1)(x-1)} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}$

$$= \frac{A(x+1)(x-1)+Bx(x-1)+Cx(x+1)}{x(x+1)(x-1)}$$

$$\frac{A(x^2-1)+B(x^2-x)+C(x^2+x)}{x(x+1)(x-1} = $$

$$= \frac{(A+B+C)x^2+(-B+C)x+(-A)}{x(x+1)(x-1)}$$

- $\frac{x^2-4}{x^4-1}$

equating coefficients $\Rightarrow$ so

① $A+B+C=0$

② $-B+C=0$ $\Rightarrow B=C \Rightarrow A+2B=0$ ①

③ $-A=1$

⇓③

$2B=1$

$A=-1$, $B=\frac{1}{2}$, $C=\frac{1}{2}$

General technique for integrating rational functions:

- if the top has greater or equal degree than the bottom, first use polynomial division to fix this;
- factor the bottom;
- split up in to partial fractions;
- integrate each.

so

$$\frac{1}{x^3-x} = \frac{-1}{x}+\frac{1}{2}\frac{1}{x+1}+\frac{1}{2}\frac{1}{x-1}$$

Wed 11th Dec 14.30 [illegible]

Examples:

- $\int \frac{x^2-4}{x^4+1}\,\mathrm{dx}$
- $\int \frac{x^4-x^2+1}{x^3-x}\,\mathrm{dx}$

$P(x)/Q(x)$

Repeated factors: If $Q(x)$ has repeated factors, e.g. $Q(x) = x^3 - 2x^2 + x = x(x-1)^2$, then we can still use partial fractions but we need a new trick:

If $(x-a)^n$ is a factor of $Q(x)$, in the partial fractions expression we should use

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_n}{(x-a)^n}$$

Example: $\int \frac{x^2+1}{x^3-2x^2+x}dx$

$$\frac{B+C}{x-1} = \frac{D}{x-1}$$

$$x^3-2x^2+x = x(x^2-2x+1)$$
$$= x(x-1)^2$$

$$\frac{x^2+1}{x(x-1)^2} \neq \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-1}$$

will not work

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

$$= \frac{(A+B)x^2 + (-2A-B+C)x + A}{x(x-1)^2}$$

$A+B=1$
$-2A-B+C=0$
$A=1$

$A=1,\ B=0,\ C=2$

$$\frac{2+3x}{x^2+1} = \frac{A+Bx}{x^2+1}$$

$$\int \frac{1}{(x-1)^2}dx = \int \frac{1}{u^2}du \qquad u = x-1 \quad \frac{du}{dx}=1$$

$$= -\frac{1}{u} + c$$

$$= \frac{-1}{x-1} + c$$

$$\frac{x^2-4}{x^4-1} = \frac{x^2-4}{(x^2+1)(x-1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1}$$

$$= \frac{(Ax+B)(x-1)(x+1) + C(x^2+1)(x+1) + D(x^2+1)(x-1)}{x^4-1}$$

$$= \frac{(A+C+D)x^3 + (B+C-D)x^2 + (-A+C+D)x + (-B+C-D)}{x^4-1}$$

equating coeffs: ① $A+C+D=0$

② $B+C-D=1$

③ $-A+C+D=0$

④ $-B+C-D=-4$

①&③ $\Rightarrow$ $A=0$ and $C+D=0$ so $C=-D$

② $\Rightarrow$ $B+2C=1$

④ $\Rightarrow$ $-B+2C=-4$

so "②+④" $4C=-3$ so $C=\frac{-3}{4}$

"②−④": $2B=5$

$B=5/2$

$A=0$ $C=-3/4$

$B=5/2$ $D=3/4$

so $$\frac{x^2-4}{x^4-1} = \frac{5/2}{x^2+1} - \frac{3/4}{x-1} + \frac{3/4}{x+1}$$

cipra

$$\int \frac{x^4 - x^2 + 1}{x^3 - x}\,dx = \int \frac{(x^3-x)(x)+1}{x^3-x}\,dx = \int\left(x + \frac{1}{x^3-x}\right)dx$$

$$= \frac{x^2}{2} + \int \frac{1}{x^3-x}\,dx$$

$$\frac{1}{x^3-x} = \frac{1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$= \frac{A(x+1)(x-1) + Bx(x-1) + Cx(x+1)}{x^3-x}$$

$$= \frac{(A+B+C)x^2 + (-B+C)x - A}{x^3-x}$$

equating coefficients $\Rightarrow$

$0x^2+0x\ 1 = (A+B+C)x^2+(-B+C)x-A$

$A+B+C = 0 \Rightarrow B+C=1$

$-B+C = 0 \Rightarrow B=C=\frac{1}{2}$

$-A = 1 \Rightarrow A=-1$

$$\frac{1}{x^3-x} = -\frac{1}{x} + \frac{1}{2}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x-1}$$

$$\int \frac{1}{x^3-x}\,dx = -\int \frac{1}{x}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$$

$$= -\ln|x| + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

$$= \ln\frac{\sqrt{|(x+1)(x-1)|}}{|x|} + C$$

$$\frac{1}{2}\ln b = \ln b^{1/2} = \ln\sqrt{b}$$

so $$\int \frac{x^2-4}{x^4-1}\,dx = \frac{5}{2}\int \frac{1}{x^2+1}\,dx - \frac{3}{4}\int \frac{1}{x-1}\,dx + \frac{3}{4}\int \frac{1}{x+1}\,dx$$

$$= \frac{5}{2}\arctan(x) - \frac{3}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + C$$

Review of Trig subs

$\int \sqrt{1-x^2}\,dx$

$x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

$= \int \overset{\sqrt{1-x^2}}{\cos\theta}\, \underbrace{\cos\theta\, d\theta}_{\Rightarrow dx}$

$= \int \cos^2\theta\, d\theta$

$= \int \frac{1+\cos 2\theta}{2}\, d\theta$

$= \int \frac{1}{2}\, d\theta + \frac{1}{2}\int \cos 2\theta\, d\theta$

$\frac{\theta}{2} + \frac{1}{4}\int \cos\phi\, 2\,d\theta \qquad \phi = 2\theta$

$\frac{1}{4}\int \cos\phi\, d\phi$

$\int 1\, dx = x + C$

$\frac{1}{4}\sin\phi = \frac{1}{4}\sin 2\theta$

$= \frac{1}{2}\cos\theta\sin\theta$

$= \frac{1}{2}x\sqrt{1-x^2}$

$\sin^2 + \cos^2 th = 1$

$\sqrt{1-x^2} = \cos\theta$

$\tan^2 + 1 = \sec^2$

$a^2\tan^2\theta + a^2 = a^2\sec^2\theta$

$(a\tan\theta)^2 + a^2 = (a\sec\theta)^2$

$\cos(2\theta) + 1 = c\, x^{2\theta}$

$\cos^2 + \sin^2 = 1$

$\cos^2 - \sin^2 = \cos 2\theta$

$2\cos^2\theta = 1 + \cos 2\theta$

$\sin^2 \neq 1 = c$

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A few useful formulae

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1) \qquad \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{1}{2}n(n+1)\right)^2$$

Newton's method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$