Partial fractions

Definition: A rational function is one of the form $\frac{P(x)}{Q(x)}$ where P and Q are polynomials.

e.g. $\frac{x^3-2x+1}{x^2-7}$, x^2, $\frac{1}{x^9+1}$

We already know how to integrate some of these.

Recall:

- $\int x^n \,\mathrm{dx} = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$
- $\int \frac{1}{x} \,\mathrm{dx} = \ln|x| + C$
- $\int \frac{1}{x^2+1} \,\mathrm{dx} = \arctan x + C$
- $\int \frac{2x}{x^2+1} \,\mathrm{dx} = \int \frac{1}{u} du = \ln|x^2+1| + C$

How about $\int \frac{1}{x^2-1} \,\mathrm{dx}$?

$$\begin{aligned}
\int \frac{1}{x^2-1} \,\mathrm{dx} &= \int \frac{1}{(x+1)(x-1)} \,\mathrm{dx} \\
&= \int \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) \mathrm{dx} \\
&= \frac{1}{2}\left(\int \frac{1}{x-1} \,\mathrm{dx} - \int \frac{1}{x+1}\right) \mathrm{dx} \\
&= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) \\
&= \ln \sqrt{\frac{|x-1|}{|x+1|}}
\end{aligned}$$

The key trick here was to recognise the rational function $\frac{1}{(x+1)(x-1)}$ as being a linear combination of simpler rational functions ("partial fractions"), namely $\frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$.

We will see that this technique, along with the integrals recalled above and polynomial division, allows us to integrate **any** rational function.

Example of using polynomial division:

$$\begin{aligned}
\int \frac{x^3+1}{x^2+1} \,\mathrm{dx} &= \int \frac{x(x^2+1) - x + 1}{x^2+1} \\
&= \int \left(x + \frac{1-x}{x^2+1}\right) \mathrm{dx} \\
&= \int \left(x + \frac{1}{x^2+1} - \frac{x}{x^2+1}\right) \mathrm{dx} \\
&= \frac{x^2}{2} + \arctan(x) - \frac{\ln|x^2+1|}{2} + C
\end{aligned}$$

Fact I: Any polynomial over the reals can be factored as a product of linear and quadratic factors.

Proof: Consequence of the "Fundamental Theorem of Algebra" - ask your 1ZC3 lecturer about it next semester.

Examples:

- $x^3 + 1 = (x + 1)(x^2 - x + 1)$
- $x^4 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$

Fact II: Given $\frac{P(x)}{Q(x)}$ with $\deg(P) < \deg(Q)$, if $Q(x)$ factors into distinct irreducible factors

$$Q(x) = q_1(x)\, q_2(x) \ldots q_n(x)$$

(each q_i linear or irreducible quadratic) then there exist $p_i(x)$ with $deg(p_i) < deg(q_i)$ such that

$$\frac{P(x)}{Q(x)} = \sum_i \frac{p_i(x)}{q_i(x)}.$$

Proof: Can be proven quite easily with linear algebra. Ask your 1ZC3 lecturer about this too!

Examples:

- $\frac{1}{x^3 - x}$

- $\frac{x^2 - 4}{x^4 - 1}$

General technique for integrating rational functions:

- if the top has greater degree than the bottom, first use polynomial division to fix this;
- factor the bottom;
- split up in to partial fractions;
- integrate each.

Examples:

- $\int \frac{x^2-4}{x^4+1}\,\mathrm{dx}$
- $\int \frac{x^4-x^2+1}{x^3-x}\,\mathrm{dx}$

Repeated factors: If $Q(x)$ has repeated factors, e.g. $Q(x) = x^3 - 2x^2 + x = x(x-1)^2$, then we can still use partial fractions but we need a new trick:

If $(x-a)^n$ is a factor of $Q(x)$, in the partial fractions expression we should use

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + ... + \frac{A_n}{(x-a)^n}$$

Example: $\int \frac{x^2+1}{x^3-2x^2+x} dx$