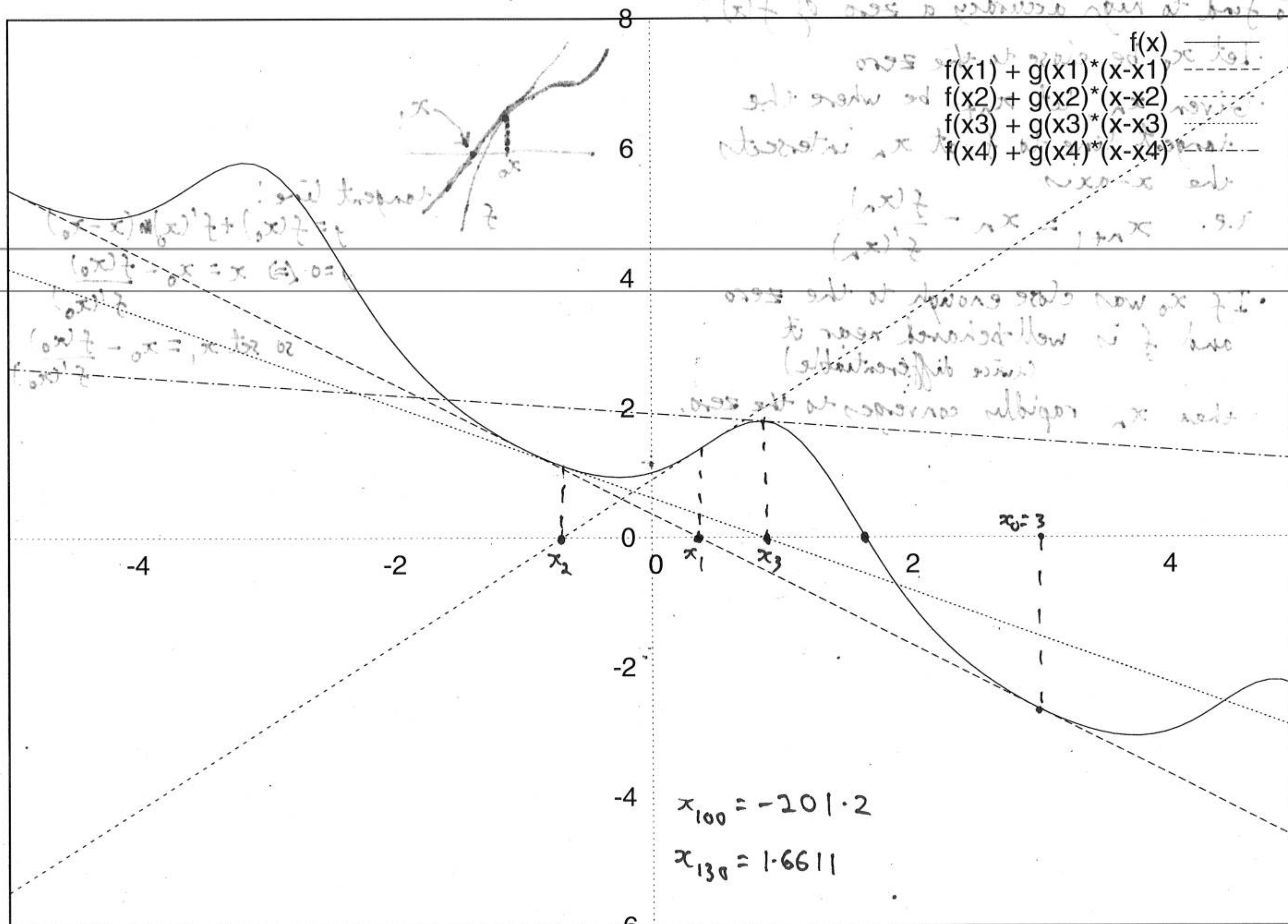
f(x)
f(x1) + g(x1)*(x-x1)
f(x2) + g(x2)*(x-x2)
f(x3) + g(x3)*(x-x3)
f(x4) + g(x4)*(x-x4)
8
6
4
2
0
-2
-4
-6
-4
-2
0
2
4
$x_0 = 3$
x_1
x_2
x_3
$x_{100} = -201\cdot 2$
$x_{130} = 1\cdot 6611$

Explanation in terms of linear approximations: Near b, $g(x) \approx g(b) + g'(b)(x-b)$ Near $g(b)$, $f(u) \approx f(g(b)) + f'(g(b))(u - g(b))$ So near b,

Not on exam

$$\left\{\begin{aligned} f(g(x)) &\approx f(g(b) + g'(b)(x-b)) \\ &\approx f(g(b)) + f'(g(b))\big(g(b) + g'(b)(x-b) - g(b)\big) \\ &= f(g(b)) + \underbrace{g'(b)f'(g(b))}(x-b) \end{aligned}\right.$$

Example:

$$\frac{d}{dx}e^{\sin(x)} = (\exp o \sin)'(x) = \sin'(x)\exp'(\sin(x)) = \cos(x)e^{\sin(x)}$$

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

Alternative notation: ~~That is,~~ if u is a function of x and y is a function of u, say $u = g(x)$ and $y = f(u) = f(g(x))$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)\,g'(x) = f'(g(x))\,g'(x)$$

Example:

$$\frac{d}{dx}(x^3-1)^9$$

$y := (x^3-1)^9$, $u := x^3 - 1$, so $y = u^9$; so

$$\frac{d}{dx}(x^3-1)^9 = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 9u^8 3x^2 = 27x^2(x^3-1)^8.$$

Differentiating invertible functions

$(f^{-1})'$

Suppose f is invertible, so $x = f^{-1}(f(x))$.

Suppose f^{-1} is differentiable. Chain rule:

$$1 = \frac{d}{dx}x = \frac{d}{dx}f^{-1}(f(x)) = f'(x)f^{-1\prime}(f(x))$$

so

$$\boxed{f^{-1\prime}(f(x)) = \frac{1}{f'(x)}.}$$

Fact: If f is invertible and is differentiable at x, then f^{-1} is differentiable at $f(x)$, and $f^{-1\prime}(f(x)) = \frac{1}{f'(x)}$.

Examples:

$$\ln'(\exp(x)) = \frac{1}{\exp'(x)} = \frac{1}{\exp(x)}$$

i.e.

$$\ln'(y) = \frac{1}{y}.$$

$$\arcsin'(\sin(x)) = \frac{1}{\cos(x)}$$

Now $cos(x) = \sqrt{1 - sin(x)^2}$, so

$$\arcsin'(y) = \frac{1}{\sqrt{1 - y^2}}$$

$$\arctan'(\tan(x)) = \cos^2(x) = \frac{1}{1 + \tan^2(x)}$$

$$\arctan'(y) = \frac{1}{1 + y^2}$$

Power rule: For t a real number,

$$\frac{d}{dx}x^t = \frac{d}{dx}e^{\ln x^t} = \frac{d}{dx}e^{t \ln x} = \frac{t}{x}e^{t \ln x} = tx^{t-1}$$

Implicit differentiation

Suppose we know some relation between x and y, e.g.

$$x^2 + y^2 = 1.$$

Here, y isn't a function of x.

But if we restrict attention to $y \geq 0$, then y is a function of x; similarly for $y \leq 0$. These functions are *implicitly* defined by $x^2 + y^2 = 1$.

Restricting to a function in this way, it makes sense to differentiate with

respect to x:

$$0 = \frac{d}{dx}1 = \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 2x + \frac{dy}{dx}2y$$

and we conclude that, whichever function we chose,

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

for all x at which the function is differentiable.

Confirm this agrees with the chain rule.

Another example: TODO