Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example:

$$e^{\sin(\frac{\pi}{2}x)} = x$$

Let $f(x) = e^{\sin(\frac{\pi}{2}x)} - x$

$$f'(x) = \frac{\pi}{2}\cos(\frac{\pi}{2}x)e^{\sin(\frac{\pi}{2}x)} - 1$$

$$\begin{aligned}
x_0 &= 2 \\
x_1 &= 1.6110154703516575 \\
x_2 &= 1.6609035259641185 \\
x_3 &= 1.6611375113211584 \\
x_4 &= 1.6611375194231848 \\
x_5 &= 1.6611375194231848
\end{aligned}$$

but

$$\begin{aligned}
x_0 = 3, x_1 &= 0.367879441171442 33 \\
x_2 &= -0.7004442394773245 \\
x_3 &= 0.8682758685413106 \\
x_4 &= 13.55349391880111 \\
\ldots x_{100} &= -201.15250823446343 \\
\ldots x_{130} &= 1.6611375194231848
\end{aligned}$$

Failure:

$$\begin{aligned}
f(x) &:= x^{1/3} \\
f'(x) &= (1/3)x^{-2/3} \\
\frac{f(x)}{f'(x)} &= 3x
\end{aligned}$$

So $x_{n+1} = -2x_n$, so diverges unless $x_0 = 0$.

Chain rule

$$(f \circ g)' = g'(f' \circ g)$$

Explanation in terms of linear approximations: Near b, $g(x) \approx g(b) + g'(b)(x-b)$ Near $g(b)$, $f(u) \approx f(g(b)) + f'(g(b))(u - g(b))$ So near b,

$$\begin{aligned} f(g(x)) &\approx f(g(b) + g'(b)(x-b)) \\ &\approx f(g(b)) + f'(g(b))(g(b) + g'(b)(x-b) - g(b)) \\ &= f(g(b)) + g'(b)f'(g(b))(x-b) \end{aligned}$$

Example:

$$\frac{d}{dx}e^{\sin(x)} = (\exp \circ \sin)'(x) = \sin'(x)\exp'(\sin(x)) = \cos(x)e^{\sin(x)}$$

Alternative notation: if u is a function of x and y is a function of u, say $u = g(x)$ and $y = f(u) = f(g(x))$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example:

$$\frac{d}{dx}(x^3-1)^9$$

$y := (x^3-1)^9$, $u := x^3 - 1$, so $y = u^9$; so

$$\frac{d}{dx}\sqrt{x^3-1} = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 9u^8 3x^2 = 27x^2(x^3-1)^8.$$

Differentiating invertible functions

Suppose f is invertible, so $x = f^{-1}(f(x))$.

Suppose f^{-1} is differentiable. Chain rule:

$$1 = \frac{d}{dx}x = \frac{d}{dx}f^{-1}(f(x)) = f'(x)f^{-1\prime}(f(x))$$

so

$$f^{-1\prime}(f(x)) = \frac{1}{f'(x)}.$$

Fact: If f is invertible and is differentiable at x, then f^{-1} is differentiable at $f(x)$, and $f^{-1\prime}(f(x)) = \frac{1}{f'(x)}$.

Examples:

$$\ln'(\exp(x)) = \frac{1}{\exp'(x)} = \frac{1}{\exp(x)}$$

i.e.

$$\ln'(y) = \frac{1}{y}.$$

$$\arcsin'(\sin(x)) = \frac{1}{\cos(x)}$$

Now $cos(x) = \sqrt{1 - sin(x)^2}$, so

$$\arcsin'(y) = \frac{1}{\sqrt{1-y^2}}$$

$$\arctan'(\tan(x)) = \cos^2(x) = \frac{1}{1+\tan^2(x)}$$

$$\arctan'(y) = \frac{1}{1+y^2}$$

Power rule: For t a real number,

$$\frac{d}{dx}x^t = \frac{d}{dx}e^{\ln x^t} = \frac{d}{dx}e^{t\ln x} = \frac{t}{x}e^{t\ln x} = tx^{t-1}$$

Implicit differentiation

Suppose we know some relation between x and y, e.g.

$$x^2 + y^2 = 1.$$

Here, y isn't a function of x.

But if we restrict attention to $y \geq 0$, then y is a function of x; similarly for $y \leq 0$. These functions are *implicitly* defined by $x^2 + y^2 = 1$.

Restricting to a function in this way, it makes sense to differentiate with

respect to x:

$$0 = \frac{d}{dx}1 = \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 2x + \frac{dy}{dx}2y$$

and we conclude that, whichever function we chose,

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

for all x at which the function is differentiable.

Confirm this agrees with the chain rule.

Another example: TODO