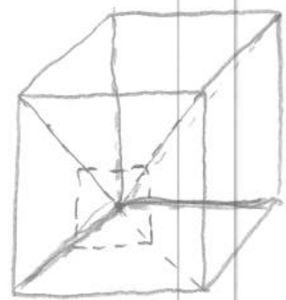


Cube = 3·Cone − 3·triangle + line

so $n^3 = 3S_n - 3T_n + n$

$3S_n = n - n^3 - \frac{3n(n+1)}{2}$

Sum of consequetive squares:

$$S_n := \sum_{i=0}^{n} i^2$$

$$\boxed{S_n = \frac{n(n+1)(2n+1)}{6}}$$

We can test this formula: clearly it works for $n = 0$, and if it works for $n = k - 1$ then

$$\begin{aligned}
S_k &= S_{k-1} + k^2 \\
&= \frac{(k-1)(k)(2k-1)}{6} + k^2 \\
&= \frac{k((k-1)(2k-1) + 6k)}{6} \\
&\cancel{= \frac{k((k-1)(2k-1) + 6k)}{6}} \\
&= \frac{k(k+1)(2k+1)}{6}.
\end{aligned}$$

(since $((k-1)+2)((2k-1)+2) = (k-1)(2k-1) + (4k-2) + (2k-2) + 4 = (k-1)(2k-1) + 6k$)

So the formula works for all n.

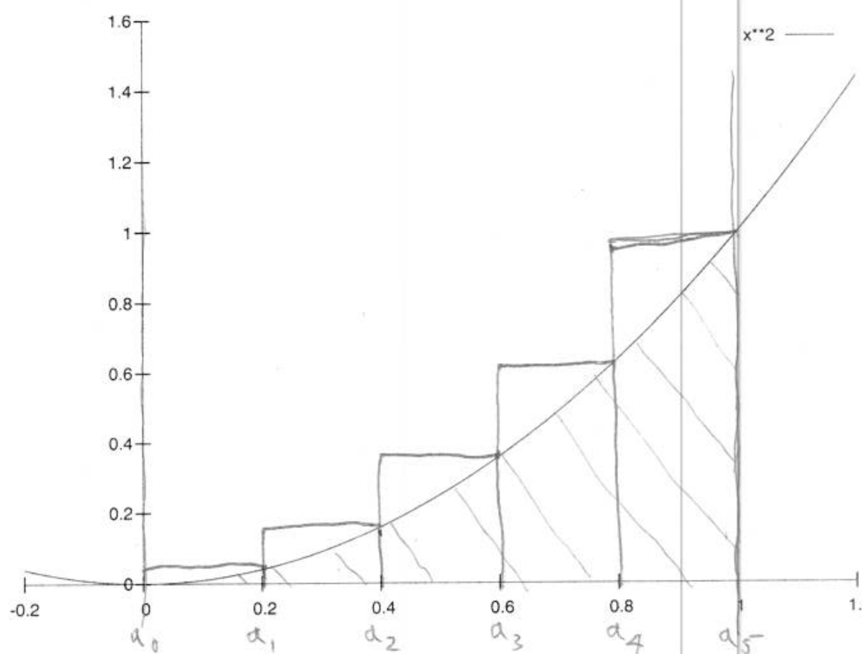
Estimating areas

Areas of shapes defined by straight lines (rectangles, triangles, polygons etc) are easy to calculate. But what about when the boundary is a curve?

e.g. What is the area of an ellipse? What is the area below a catenary?

Area beneath a graph: Let $[a, b]$ be an interval and let $f(x)$ be a function continuous and non-negative on the interval. We will try to estimate the area bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.

e.g. $f(x) = x^2$, $[a, b] = [0, 1]$.



Idea: estimate area below the graph as the sum of the areas of rectangles, with height given by evaluating the function. When width of the rectangles is small, this should be a good estimate.

e.g. split $[0, 1]$ into n equally sized intervals, so the endpoints are $a_i = i/n$ for $i = 0, 1, ..., n$, and consider n rectangles with bases these intervals, and with height the value of the function at, say, the right end-point of the corresponding interval.

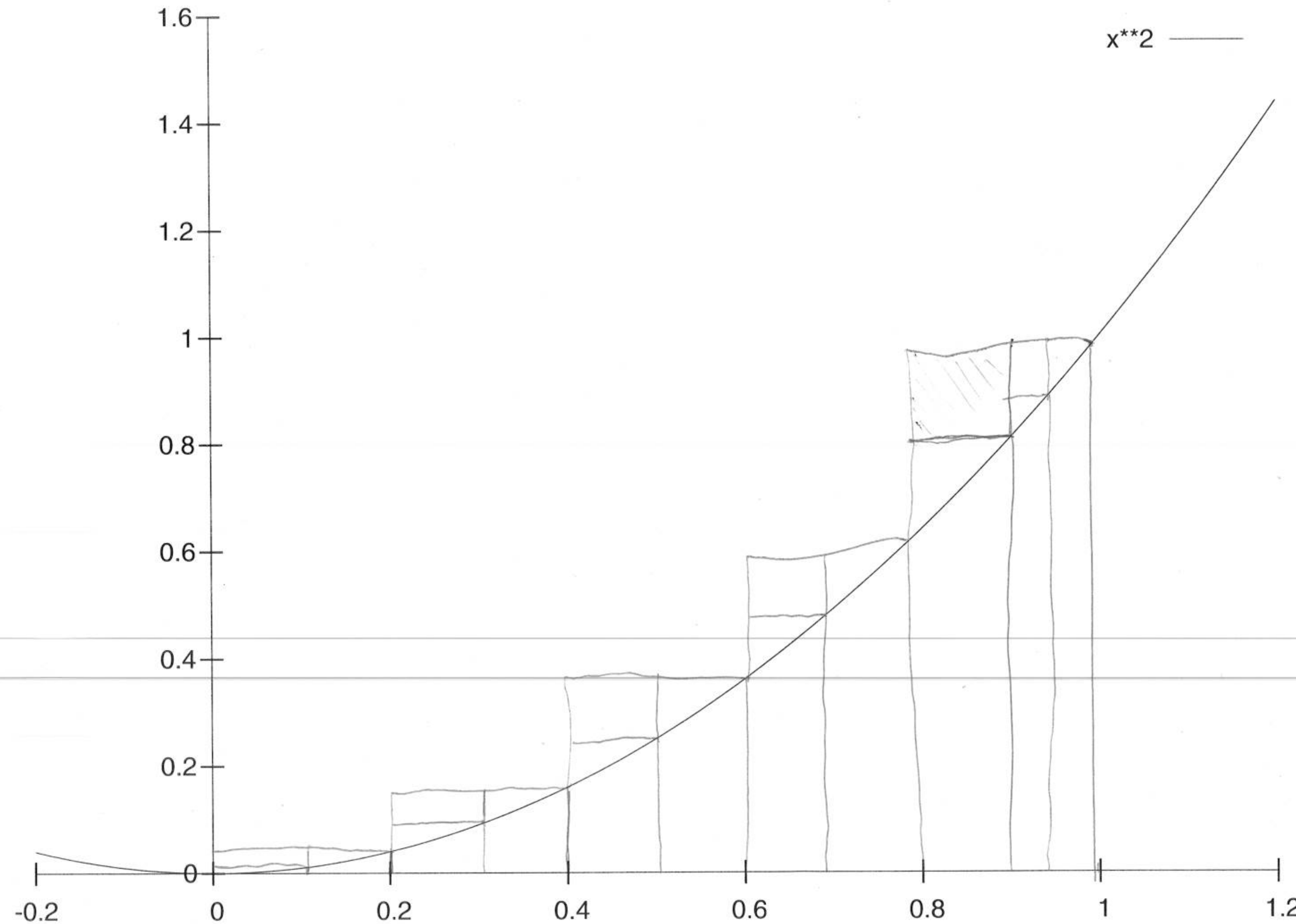
So the i^{th} rectangle has width $1/n$ and height $f(a_i) = f\left(\frac{i}{n}\right) = \left(\frac{i}{n}\right)^2$, so its area is

$$RectArea_i = \left(\frac{1}{n}\right)\left(\frac{i}{n}\right)^2 = \frac{i^2}{n^3}.$$

So the sum of the areas is

$$\begin{aligned} A_n &= \sum_{i=1}^{n} RectArea_i \\ &= \sum_{i=1}^{n} \frac{i^2}{n^3} \qquad = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6n^3}. \quad = \frac{2n^3 + \dots}{6n^3} \end{aligned}$$

$$\lim_{n\to\infty} A_n = \frac{1}{3}$$


x**2
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2

e.g. with $n = 10$: $A_{10} = 10 * 11 * 21/6000 = 0.385$. with $n = 1000$: $A_{1000} = (1000 * 1001 * 2001) / (6 * 1000 * 1000 * 1000) = 0.3338335$

Now: since the estimate gets more and more accurate for larger n, we can expect that the area *is* the limit $\lim_{n\to\infty} A_n = \frac{1}{3}$.

Remarks: It wasn't important to our reasoning that we took the value of f at the right end-point of each interval to define the height of the corresponding rectangle. Taking the value of f at *any* point of the interval should work just as well.

Sometimes, we won't be able to find a nice formula for the limit as $n \to \infty$ as we could above. Still, we expect the above approach to give a good estimate (assuming f is "reasonable").

Definite Integrals

Definition: A function f is integrable on an interval $[a, b]$ if the limit $\lim_{n\to\infty} S_n$ of Riemann sums exists and is the same for any choice of Riemann sums, and in this case that limit is the definite integral of f from a to b.

Here, a Riemann sum S_n is the sum

$$S_n = \sum_{i=1}^{n} \Delta_n f(x_i^*)$$

length of little interval $= \Delta_n = \frac{b-a}{n}$

where $\Delta_n = \frac{b-a}{n}$, and x_i^* is a choice of a point in the interval

i^{th} interval $[a + (i-1)\Delta_n, a + i\Delta_n]$.

So the definite integral is the limit of Riemann sums; but if f is ill-behaved, this limit might depend on exactly how we calculate the Riemann sums (what points we calculate f at), so then we don't get a well-defined integral and we say that f is not integrable on $[a, b]$. Luckily...

Theorem: If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.