Adding up

Summation notation: Given numbers a_i and integers $m \leq n$,

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + ... + a_n.$$

So e.g. if we have 50 boxes containing balls, and the i^{th} box contains B_i balls, then the total number of balls in the boxes is

$$\sum_{i=1}^{50} B_i.$$

Similarly, given a function f and integers $m \leq n$,

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + ... + f(n).$$

So e.g. we can abbreviate the sum

$$1 + 2 + ... + n$$

as

$$\sum_{i=1}^{n} i.$$

Remarks: By basic algebra,

$$\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$$

$$\sum_{i=m}^{n} (a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right).$$

Also, if $m \leq n < s$, then

$$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{s} a_i = \sum_{i=m}^{s}.$$

We don't have to use i as the index; e.g.

$$\sum_{n=s}^{t} f(n) = \sum_{i=s}^{t} f(i)$$

Examples:

$$\sum_{i=n}^{n} a_i = a_n$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=0}^{n} 1 = n + 1$$

$$\sum_{i=n}^{n} 1 = 1$$

$$\sum_{i=m}^{n} 1 = n + 1 - m$$

The n^{th} /triangular number/ is

$$T_n := \sum_{i=0}^{n} i.$$

$$\begin{aligned} 2T_n &= \sum_{i=0}^{n} i + \sum_{i=0}^{n} (n - i) \\ &= \sum_{i=0}^{n} n \\ &= (n+1)\, n. \end{aligned}$$

So

$$T_n = \frac{n\,(n+1)}{2}.$$

Note that we can then easily calculate e.g.

$$\begin{aligned} \sum_{i=37}^{1337} i &= \sum_{i=0}^{1337} i - \sum_{i=0}^{36} \\ &= T_{1337} - T_{36} \\ &= \frac{(1337)\,(1338) - (36)\,(37)}{2} \\ &= 893787. \end{aligned}$$

Sum of consequetive squares:

$$S_n := \sum_{i=0}^{n} i^2$$

$$S_n = \frac{n\,(n+1)\,(2n+1)}{6}$$

We can test this formula: clearly it works for $n = 0$, and if it works for $n = k - 1$ then

$$\begin{aligned} S_k &= S_{k-1} + k^2 \\ &= \frac{(k-1)\,(k)\,(2k-1)}{6} + k^2 \\ &= \frac{k\,((k-1)\,(2k-1) + 6k)}{6} \\ &= \frac{k\,((k-1)\,(2k-1) + 6k)}{6} \\ &= \frac{k\,(k+1)\,(2k+1)}{6}. \end{aligned}$$

(since $((k-1)+2)\,((2k-1)+2) = (k-1)\,(2k-1) + (4k-2) + (2k-2) + 4 = (k-1)\,(2k-1) + 6k$)
So the formula works for all n.

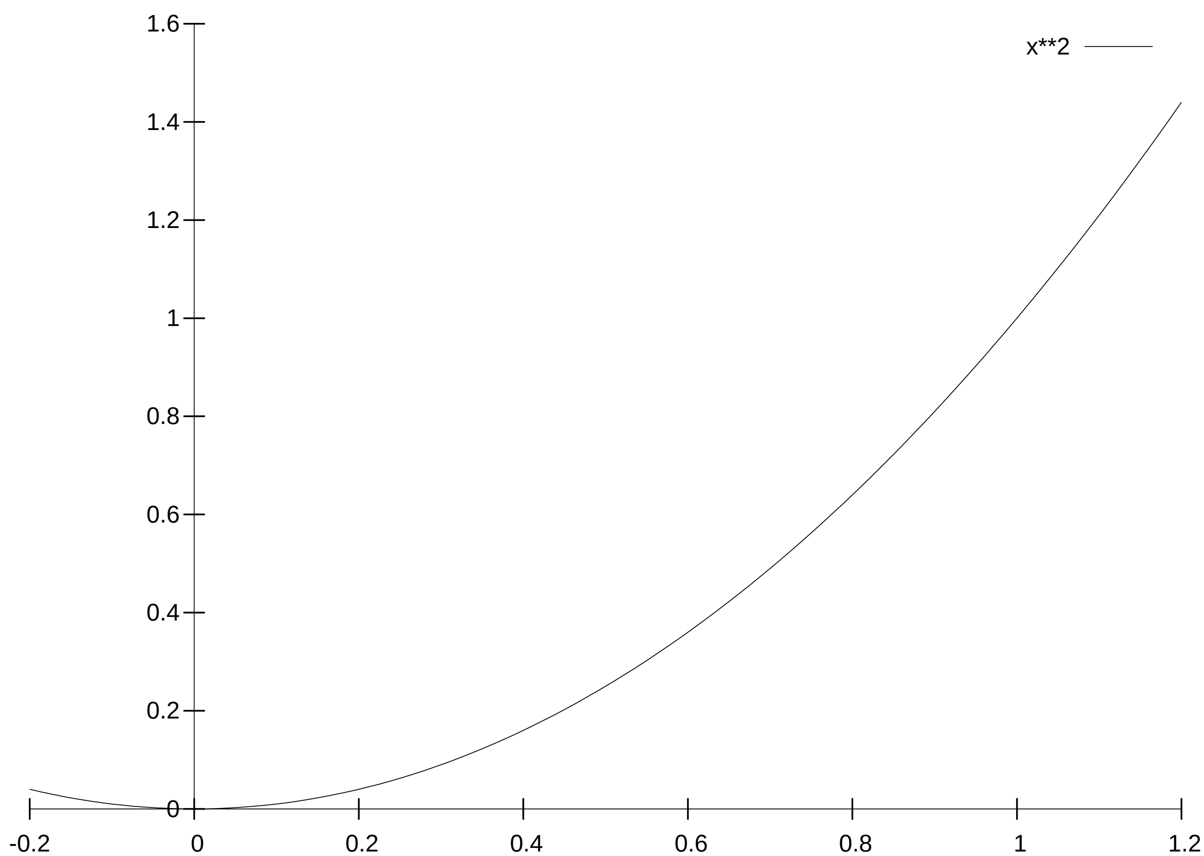
Estimating areas

Areas of shapes defined by straight lines (rectangles, triangles, polygons etc) are easy to calculate. But what about when the boundary is a curve?

e.g. What is the area of an ellipse? What is the area below a catenary?

Area beneath a graph: Let $[a, b]$ be an interval and let $f(x)$ be a function continuous and non-negative on the interval. We will try to estimate the area bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.

e.g. $f(x) = x^2$, $[a, b] = [0, 1]$.



Idea: estimate area below the graph as the sum of the areas of rectangles, with height given by evaluating the function. When width of the rectangles is small, this should be a good estimate.

e.g. split $[0, 1]$ into n equally sized intervals, so the endpoints are $a_i = i/n$ for $i = 0, 1, ..., n$, and consider n rectangles with bases these intervals, and with height the value of the function at, say, the right end-point of the corresponding interval.

So the i^{th} rectangle has width $1/n$ and height $f(a_i) = f\left(\frac{i}{n}\right) = \left(\frac{i}{n}\right)^2$, so its area is

$$RectArea_i = \left(\frac{1}{n}\right)\left(\frac{i}{n}\right)^2 = \frac{i^2}{n^3}.$$

So the sum of the areas is

$$A_n = \sum_{i=1}^{n} RectArea_i$$

$$= \sum_{i=1}^{n} \frac{i^2}{n^3} \qquad\qquad = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{n\,(n+1)\,(2n+1)}{6n^3}.$$

e.g. with $n = 10$: $A_{10} = 10{*}11{*}21/6000 = 0.385$. with $n = 1000$: $A_{1000} = (1000 * 1001 * 2001)\,/\,(6 * 1000 * 100$
0.3338335

Now: since the estimate gets more and more accurate for larger n, we can expect that the area *is* the limit $\lim_{n\to\infty} A_n = \frac{1}{3}$.

Remarks: It wasn't important to our reasoning that we took the value of f at the right end-point of each interval to define the height of the corresponding rectangle. Taking the value of f at *any* point of the interval should work just as well.

Sometimes, we won't be able to find a nice formula for the limit as $n \to \infty$ as we could above. Still, we expect the above approach to give a good estimate (assuming f is "reasonable").

Definite Integrals

Definition: A function f is integrable on an interval $[a, b]$ if the limit $\lim_{n\to\infty} S_n$ of Riemann sums exists and is the same for any choice of Riemann sums, and in this case that limit is the definite integral of f from a to b.

Here, a Riemann sum S_n is the sum

$$S_n = \sum_{i=1}^{n} \Delta_n f\left(x_i^*\right)$$

where $\Delta_n = \frac{b-a}{n}$, and x_i^* is a choice of a point in the interval

$$[a + (i-1)\,\Delta_n, a + i\Delta_n].$$

So the definite integral is the limit of Riemann sums; but if f is ill-behaved, this limit might depend on exactly how we calculate the Riemann sums (what points we calculate f at), so then we don't get a well-defined integral and we say that f is not integrable on $[a, b]$. Luckily...

Theorem: If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Notation: We write

$$\int_a^b f(x)\,dx$$

for the definite integral from a to b of f.

"dx" here should be read as notation indicating the variable we are integrating with respect to, much like the $\frac{d}{dx}$ of differentiation. So e.g.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

So
$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\frac{(b-a)}{n}\sum_{i=1}^{n} f(x_i^*)\right)$$
where for each n, each x_i^* is a choice of point in the n^{th} interval, and the limit exists and doesn't depend on these choices (which is true if f is continuous on $[a,b]$).

So e.g. we saw above that
$$\int_0^1 x^2 dx = \frac{1}{3}$$

If f is non-negative on $[a,b]$, then $\int_a^b f(x)\,dx$ is precisely the limit of the estimates to the area beneath the graph we discussed above. We *define* that area to be the integral. More generally:

Interpretation/Definition: If $a \le b$, the signed area (or net area) between the graph of f, the x-axis, and the vertical lines $y=a$ and $y=b$ is defined to be $\int_a^b f(x)\,dx$.

So the signed area is the sum of the areas below the positive parts of the graph minus the sum of the areas above the negative parts.

Example:
$$\int_{-2}^{2}\left(x^3 - x\right)dx$$

We can use right-hand endpoints, i.e. choosing sample point x_i^* to be $-2 + i\Delta_n$

$$\begin{aligned}
\int_{-2}^{2}\left(x^3-x\right)dx &= \lim_{n\to\infty}\Delta_n\sum_{i=1}^{n} f(x_i^*)\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n} f\left(-2+\frac{4i}{n}\right)\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(-2+\frac{4i}{n}\right)^3 - \left(-2+\frac{4i}{n}\right)\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(-6+\frac{(3)(4)(4i)-4i}{n}+\frac{(3)(-2)(4i)^2}{n^2}+\frac{(4i)^3}{n^3}\right)\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(-6+44\frac{i}{n}-96\frac{i^2}{n^2}+64\frac{i^3}{n^3}\right)\\
&= \lim_{n\to\infty}4\left(-6+44\frac{n(n+1)}{2n^2}-96\frac{n(n+1)(2n+1)}{6n^3}+64\frac{(n(n+1))^2}{4n^4}\right)\\
&= 4\left(-6+\frac{44}{2}-96\frac{1}{3}+64\frac{1}{4}\right)\\
&= 4(-6+22-32+16)\\
&= 0
\end{aligned}$$

(we used here the formula
$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 = \left(\frac{n(n+1)}{2}\right)^2$$
see Appendix E problem 40 for a rather nice proof.)

Facts:

(i) $\int_a^b 1dx = b - a$

(ii) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(iii) $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(iv) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(v) $\int_a^a f(x)\,dx = 0$

Remark: It follows from (iv) and (v) that $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ so in terms of the signed area interpretation, taking the endpoints the "wrong way round" introduces a minus sign.