Lecture 23 The fundamental theorem of calculus.

(Dr Bays class)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \qquad \Delta x = \frac{b-a}{n}$$

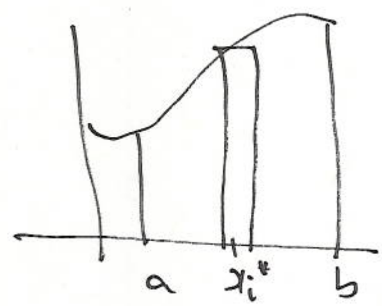
$$\int_b^a f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\frac{a-b}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)(-1)\frac{(b-a)}{n}$$

$$= -1 \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\frac{b-a}{n}$$

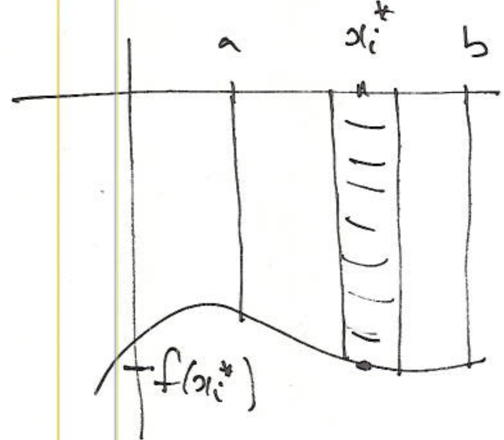
$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

If $f(x) \geq 0$ on $[a, b]$ then



$\int_a^b f(x)\,dx =$ area under curve $y = f(x)$ as x goes from a to b.

If $f(x) \le 0$ on $[a,b]$ then



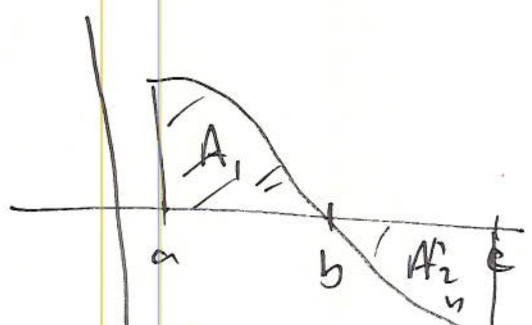
area of a rectangle

$$= -1\, f(x_i^*)\,\Delta x$$

whole area $\approx \sum_{i=1}^{n} (-1)\, f(x_i^*)\,\Delta x$

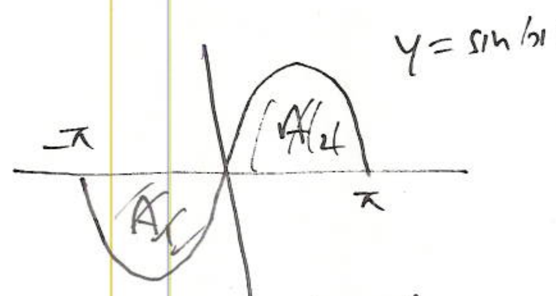
$$\text{area} = \int_a^b -1\, f(x_i^*)\,dx = -\int_a^b f(x)\,dx.$$

If $f(x) \ge 0$ on $[a,b]$ and $f(x) \le 0$ on $[b,c]$



$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$= A_1 - A_2.$$

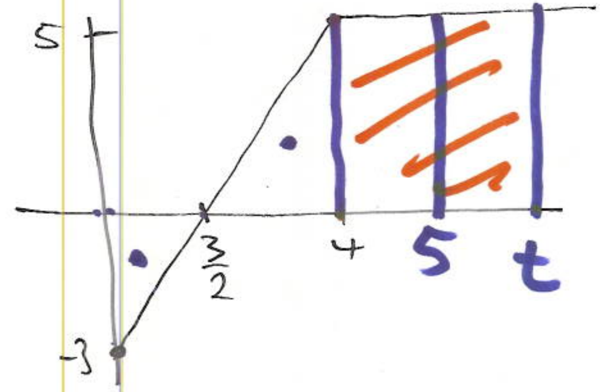


Ex. $$\int_{-\pi}^{\pi} \sin(x)\,dx = -A_1 + A_2$$

$$= 0.$$

$A_1 = A_2$

Ex $\quad f(x) = \begin{cases} 2x-3, & 0 \le x \le 4 \\ 5, & x \geq 4 \end{cases}$



$$\int_0^5 f(x)\,dx = \int_0^{3/2} f(x)\,dx + \int_{3/2}^{4} f(x)\,dx + \int_4^5 f(x)\,dx$$

$$= - \triangledown + \triangle + \square$$

$$= -\frac{1}{2} 3 \cdot \frac{3}{2} + \frac{1}{2}\left(4 - \frac{3}{2}\right)5 + 5 \cdot 1$$

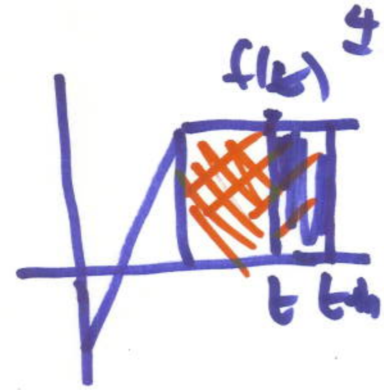
$$= -\frac{9}{4} + \frac{25}{4} + 5$$

$$= 4 + 5 = 9.$$

Write $g(t) = \int_0^t f(x)\,dx \quad , \quad t \geq 4$

$$= \int_0^4 f(x)\,dx + \int_4^t f(x)\,dx$$

$$g(t) = 4 + (t-4)5.$$



$$g'(t) = \lim_{h\to 0} \frac{g(t+h)-g(t)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\int_0^{t+h} f(x)\,dx - \int_0^{t} f(x)\,dx\right)$$

$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left(f(t)\cancel{h}\right)$$

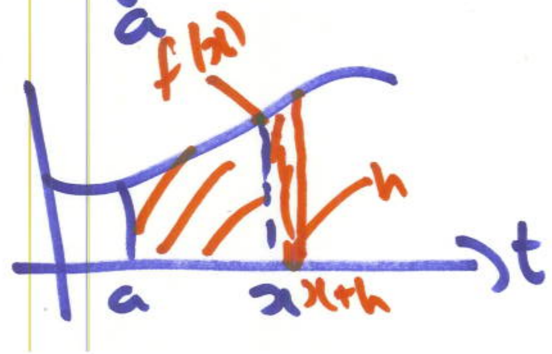
$$= f(t)$$

g is an antiderivative of f.

<u>Fundamental Thm of Calculus - Part I</u>.

Let f be a function which is continuous on $[a,b]$. Define $g(x) = \int_a^x f(t)\,dt$.

Then g is continuous on $[a,b]$, differentiable



on (a,b) and

$$g'(x) = f(x).$$

Thus g is an antiderivative of the function f.

Let F be any other antiderivative of f.

Then $F(x) = g(x) + C$, for some constant C.

So $$F(b) = g(b) + C$$
$$F(a) = g(a) + C$$

$$F(b) - F(a) = g(b) + \cancel{C} - g(a) - \cancel{C}$$
$$= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$$

$$F(b) - F(a) = \int_a^b f(t)\,dt - 0$$

FTC Part II.

Let F be any antiderivative of f.

Then $\displaystyle\int_a^b f(t)\,dt = F(b) - F(a)$.

Ex. $\displaystyle\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1$

$= \frac{1}{3}1^3 - \frac{1}{3}0^3$

$= \frac{1}{3}$.