Fundamental Theorem of Calculus (FTC) Reiterated

Theorem [FTC]:

(I) Let f be continuous on $[a, b]$. Then

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable on (a, b) and continuous on $[a, b]$, and for x in (a, b) we have

$$F'(x) = f(x).$$

So for continuous f,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

(II) If f is continuous on $[a, b]$ and $F' = f$ on $[a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

In other words: if g is differentiable on $[a, b]$ with continuous derivative, then

$$\int_a^b g'(t)\,dt = [g(t)]_a^b = g(b) - g(a).$$

Idea of proof:

(I)

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) \\
&= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \\
&= \lim_{h\to 0} \frac{1}{h}[\text{signed area xetween t=x and t=x+h}] \\
&\qquad \approx hf(x) \\
&\qquad\qquad \text{(since } f \text{ is continuous at } x \text{ and integrable around } x\text{)} \\
&= f(x)
\end{aligned}$$

(II) By (I), $\int_a^x g'(t)\,dt$ is an antiderivative of $g'(t)$ on (a, b). So by the MVT, $g(x) = \int_a^x g'(t)\,dt + C$ on (a, b), and hence by continuity on $[a, b]$.

Since $\int_a^a g'(t)\,dt = 0$, we must have $C = g(a)$. So

$$\int_a^b g'(t)\,dt = g(b) - g(a).$$

Note that FTC-I makes sense of the indefinite integral notation for antiderivatives, since

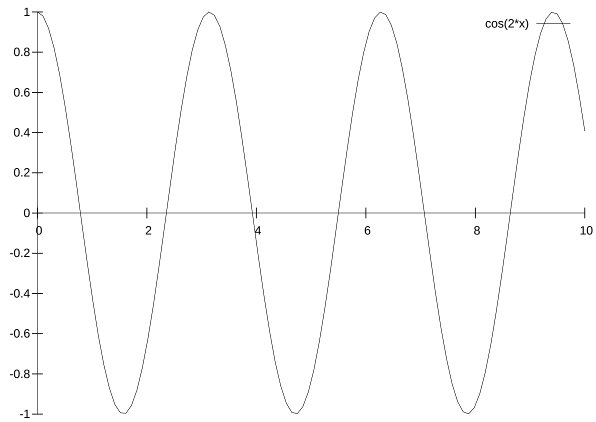
$$\frac{d}{dx}\int f(x)\,dx = f(x),$$

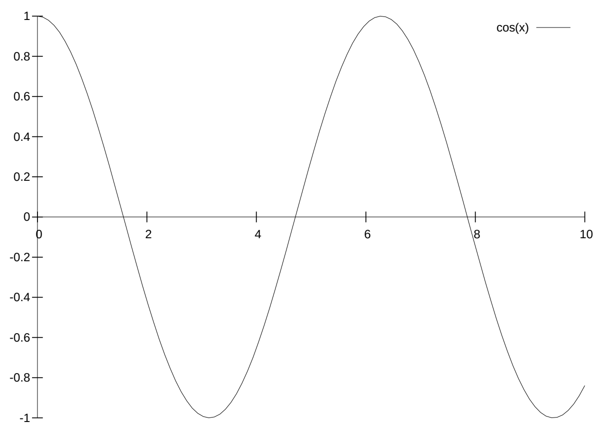
Substitution

Examples and intuition

We know $\int_2^3 \cos(x)\,dx = \sin(3) - \sin(2)\,(= -0.768)$.

Consider $\int_2^3 \cos(2x)$.





Let $u = 2x$. The area between $x = 2$ and $x = 3$ for $\cos(2x)$ corresponds to the area between $u = 4$ and $u = 6$ for $\cos(u)$ - but the former is squashed by a constant factor of 2 relative to the latter.

We can compensate for the squashing by multiplying $\cos(2x)$ by a constant factor of 2, so we expect:

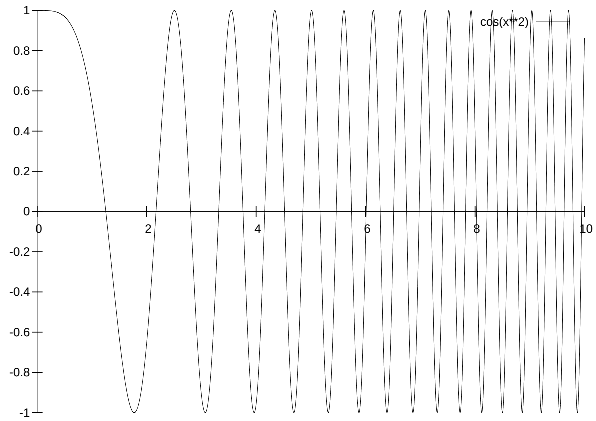
$$\int_2^3 2\cos(2x)\,dx = \int_4^6 \cos(u)\,du = \sin(6) - \sin(4)\,.$$

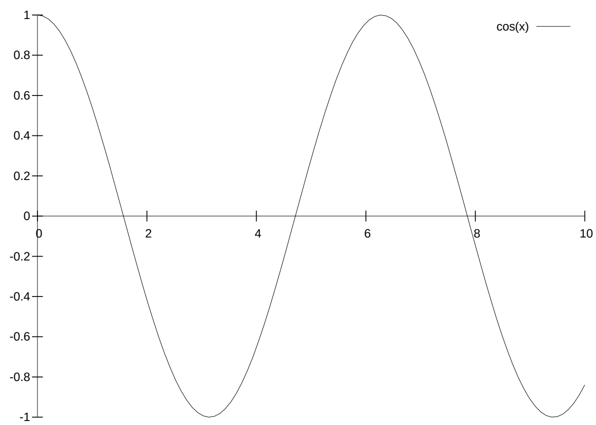
Indeed, by the chain rule, $\sin(2x)$ is an antiderivative of $2\cos(2x)$, so this is right.

We deduce

$$\int_2^3 \cos(2x)\,dx = \frac{1}{2}\int_2^3 2\cos(2x)\,dx = \frac{\sin(6) - \sin(4)}{2} = 0.239$$

Now consider $\int_2^3 \cos(x^2)$.





Let $u = x^2$. The area between $x = 2$ and $x = 3$ for $\cos(x^2)$ corresponds to the area between $u = 4$ and $u = 9$ for $\cos(u)$ - but the former is squashed by a factor of $\frac{d}{dx}x^2 = 2x$ relative to the latter.

We can compensate for the squashing by multiplying $\cos(x^2)$ by a factor of $2x$, so we expect:

$$\int_2^3 2x \cos\left(x^2\right) dx = \int_4^9 \cos(u)\, du = \sin(9) - \sin(4)\,.$$

Again, we can confirm this using the chain rule: $\sin(x^2)$ is an antiderivative of $2x\cos(x^2)$.

Note that we have actually discovered **nothing** about $\int_2^3 \cos(x^2)$! Instead, we have found an entirely **different** integral, namely

$$\int_2^3 2x \cos\left(x^2\right) dx.$$

There is no way to get from that to any information about $\int_2^3 \cos(x^2)$!

Formal formulation

Theorem [substitution rule]:

(a) For indefinite integrals: Suppose f is continuous and g is differentiable. Then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

where $u = g(x)$.

(b) For definite integrals: Suppose further that g' is continuous. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof:

(a) If F is an antiderivative of f, then by the chain rule

$$\frac{d}{dx} F(g(x)) = f(g(x))\, g'(x)$$

so $F(u) = F(g(x))$ is an antiderivative of $f(g(x))\, g'(x)$.

(b) Now by FTC-II

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\, du$$

Further examples

$$\begin{aligned}
\int_{-1}^{2} x^3 e^{x^4} dx &= \frac{1}{3}\int_{-1}^{2} e^{x^4} 3x^3 dx \\
&= \frac{1}{3}\int_{(-1)^4}^{2^4} e^u du && \left(u = x^4, \frac{du}{dx} = 4x^3\right) \\
&= \left[\frac{1}{3}e^u\right]_{(-1)^4}^{2^4} \\
&= \left[\frac{1}{3}e^u\right]_{1}^{16} \\
&= \frac{e^{16} - e}{3} \\
&= 2.96 * 10^6
\end{aligned}$$

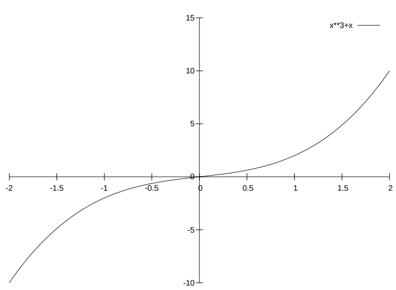
$$\begin{aligned}
\int x^3 e^{x^4} dx &= \frac{1}{3}\int e^{x^4} 3x^3 dx \\
&= \frac{1}{3}\int e^u du && \left(u = x^4, \frac{du}{dx} = 4x^3\right) \\
&= \frac{1}{3}e^u + C \\
&= \frac{e^{x^4}}{3} + C
\end{aligned}$$

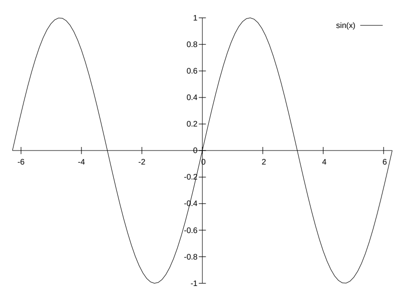
$$\begin{aligned}\int \frac{1}{\sqrt{2+3x}}dx &= \frac{1}{3}\int \frac{1}{\sqrt{2+3x}}3dx \\ &= \frac{1}{3}\int \frac{1}{\sqrt{u}}du && \left(u = 2+3x, \frac{du}{dx} = 3\right) \\ &= \frac{1}{3}2\sqrt{u} + C \\ &= \frac{2}{3}\sqrt{2+3x} + C\end{aligned}$$

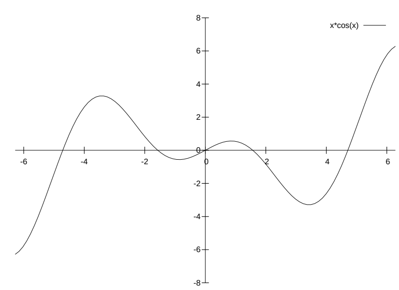
$$\begin{aligned}\int_1^e \frac{2x+\ln x}{x}dx &= \int_1^e \frac{2x}{x}dx &&+ \int_1^e \frac{\ln x}{x}dx \\ &= \int_1^e 2dx &&+ \int_0^1 udu \qquad \left(u = \ln x, \frac{du}{dx} = \frac{1}{x}\right) \\ &= [x]_{x=1}^{x=e} &&+ \left[\frac{u^2}{2}\right]_{u=0}^{u=1} \\ &= e-1 &&+ \frac{1}{2} - 0 \\ &= e - \frac{1}{2}\end{aligned}$$

Exploiting symmetry

Suppose $f(x)$ is an odd function, i.e. $f(-x) = -f(x)$



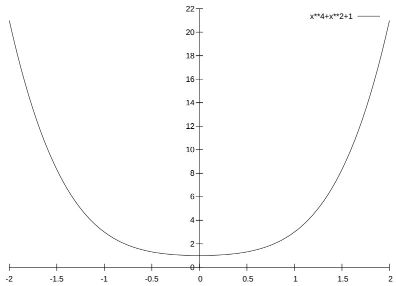


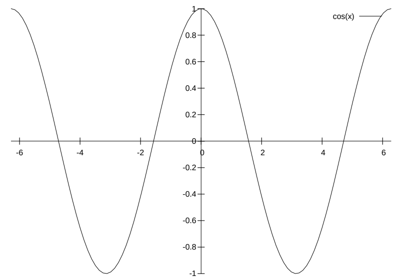


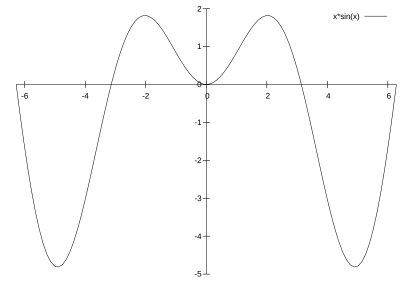
Then

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \\ &= \int_{-a}^{0} -f(-x)\,dx + \int_0^a f(x)\,dx \\ &= \int_a^0 f(u)\,du + \int_0^a f(x)\,dx && \left(u = -x, \frac{du}{dx} = -1\right) \\ &= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx && = 0\end{aligned}$$

Similarly, if $f(x)$ is an even function, i.e. $f(-x) = f(x)$







Then

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{-a}^{0} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx \qquad \left(u = -x, \frac{du}{dx} = -1\right) \\ &= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx\end{aligned}$$

$ln|x|$ as an antiderivative of $\frac{1}{x}$

Recall that $\frac{d}{dx}lnx = \frac{1}{x}$. So e.g. it does follow that $\int_1^2 \frac{1}{x}dx = ln2 - ln1$.

But lnx is only defined for $x > 0$, while $\frac{1}{x}$ is also defined for $x < 0$.

Cunning trick: When $x > 0$:

$$\frac{d}{dx}ln|x| = \frac{d}{dx}lnx = \frac{1}{x}$$

When $x < 0$:

$$\frac{d}{dx}ln|x| = \frac{d}{dx}ln(-x) = -\frac{1}{-x} = \frac{1}{x}$$

So we can write

$$\int \frac{1}{x}dx = \ln|x| + C,$$

(meaning that this is family of all antiderivatives when we restrict to an interval not containing 0)

Warning: $\frac{1}{x}$ is **not** integrable on any interval containing 0. So e.g.

$$\int_{-1}^{1} \frac{1}{x}dx$$

does not exist (and in particular is **not** equal to $\ln|1| - \ln|-1| = 0$, even though the function is odd!).

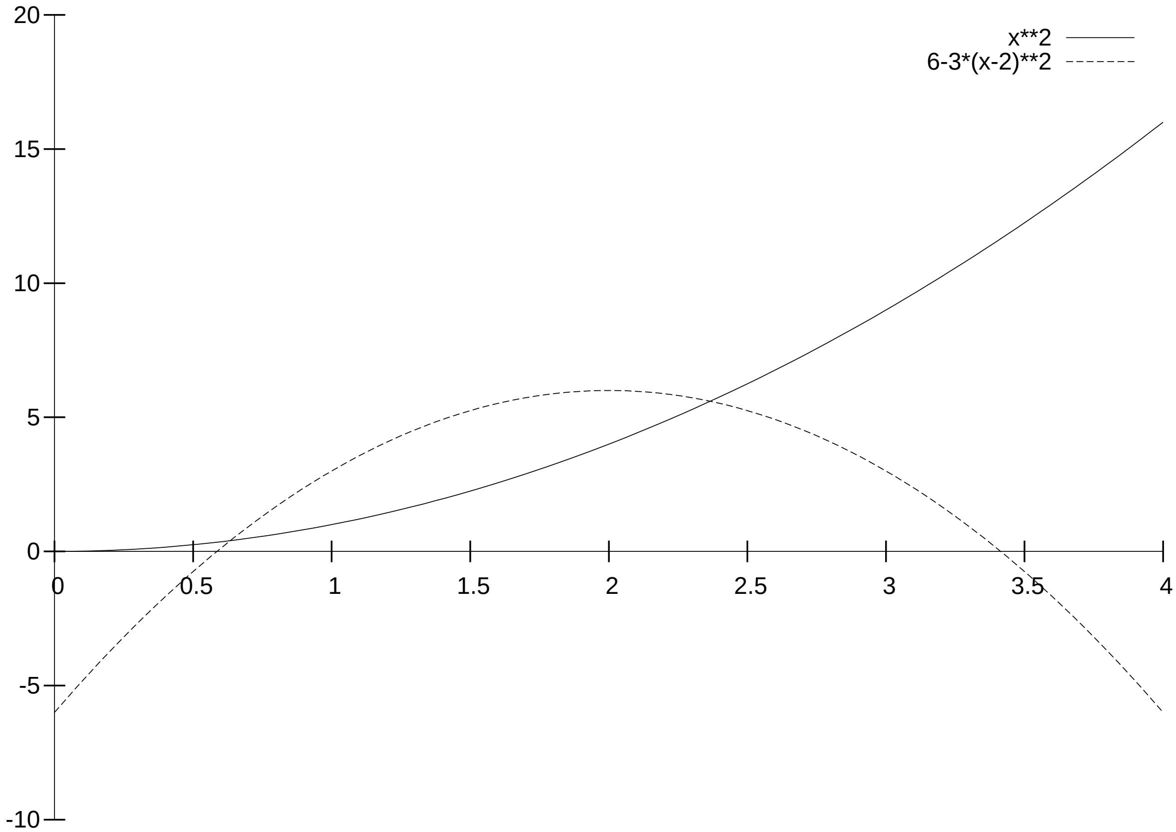
Example:

$$\begin{aligned}\int \tan(x)\,dx &= \int \frac{\sin(x)}{\cos(x)} dx \\ &= -\int \frac{1}{u}\frac{du}{dx} dx \qquad \left(u = \cos(x)\,, \frac{du}{dx} = -\sin(x)\right) \\ &= -\int \frac{1}{u} du \\ &= -\ln|u| + C \\ &= -\ln|cos(x)| + C\end{aligned}$$

(but again, you can only integrate $\tan(x)$ on intervals on which it is defined!)

Area between curves

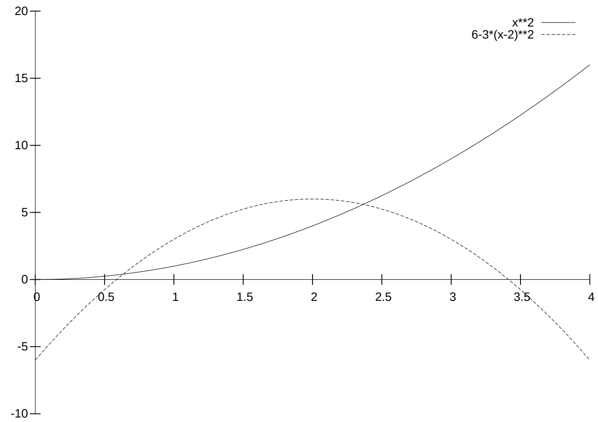
Example: Find the area of the region enclosed by the graphs of x^2 and $6-3\,(x-2)^2$.



Solution: We first find the x-values of the intersection points :

$$\begin{aligned} & x^2 = 6-3\,(x-2)^2 \\ \Leftrightarrow\ & 4x^2-12x+6=0 \\ \Leftrightarrow\ & x=\frac{6\pm\sqrt{12}}{4} \\ \Leftrightarrow\ & x=0.634 \text{ or } x=2.37 \end{aligned}$$

Then the area between the graphs is the difference between the area between top one and the x-axis and the area between the bottom one and the x-axis. So the area is



$$\int_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}} \left(6-3\left(x-2\right)^2\right) dx - \int_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}} x^2 dx = \int_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}} \left(6-3\left(x-2\right)^2 - x^2\right) dx$$

$$= \int_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}} \left(-4x^2+12x-6\right) dx$$

$$= \left[-\frac{4x^3}{3}+6x^2-6x\right]_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}}$$

$$= 3.46$$

It wasn't important that the area was above the x-axis, and so we get in general:

Formula: If f and g are continuous functions on $[a,b]$, and if $f\left(x\right) \geq g\left(x\right)$ on $[a,b]$, then the area of the region enclosed by the graphs of $f\left(x\right)$, $g\left(x\right)$ and the lines $x=a$ and $x=b$ is

$$\int_a^b \left(f\left(x\right)-g\left(x\right)\right) dx$$

Midterm remarks: Review sessions **tonight** - see yellow website.
No table of integrals will be given: learn the table on p398 of Stewart.
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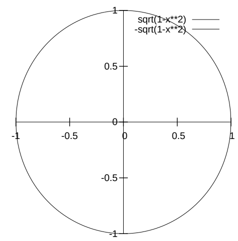


Examples: Area of the unit circle: We can think of this as the area enclosed by the graphs of $\sqrt{1-x^2}$ and $-\sqrt{1-x^2}$. So:

$$\begin{aligned}
C &= \int_{-1}^{1} \left(\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right) dx \\
&= 2\int_{-1}^{1} \sqrt{1-x^2} dx \\
&= 2\int_{-1}^{1} \sqrt{1-\cos\left(\arccos\left(x\right)\right)^2} dx \\
&= 2\int_{-1}^{1} \sqrt{1-\cos\left(\theta\right)^2} dx \qquad \left(\theta = \arccos\left(x\right); \frac{d\theta}{dx} = -\frac{1}{\sqrt{1-x^2}} = \frac{1}{-\sin\left(\theta\right)}\right) \\
&= 2\int_{-1}^{1} \sin\left(\theta\right) dx \\
&= 2\int_{-1}^{1} -\sin^2\left(\theta\right) \frac{1}{-\sin\left(\theta\right)} dx \\
&= 2\int_{\pi}^{0} -\sin^2\left(\theta\right) d\theta \\
&= 2\int_{0}^{\pi} \sin^2\left(\theta\right) d\theta \\
&= \int_{0}^{\pi} \left(1-\cos\left(2\theta\right)\right) d\theta \\
&= \left(\left[\theta\right]_0^\pi - \frac{1}{2}\left[\sin\left(2\theta\right)\right]_0^\pi\right) \\
&= \pi - 0 \\
&= \pi
\end{aligned}$$

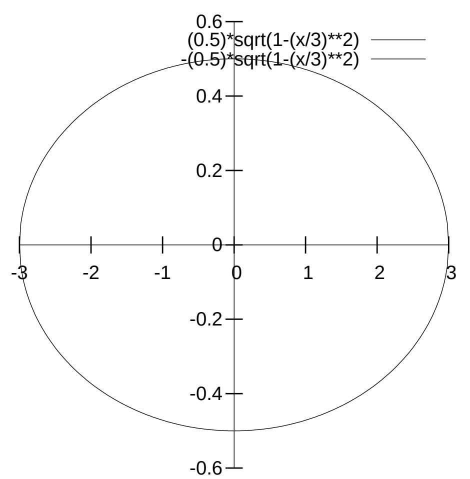
Area of an ellipse: After rotating and translating to the origin, any ellipse can be represented as the solutions to $ax^2 + by^2 = 1$, i.e.

$$y = \pm \frac{1}{\sqrt{b}} \sqrt{1 - ax^2}.$$

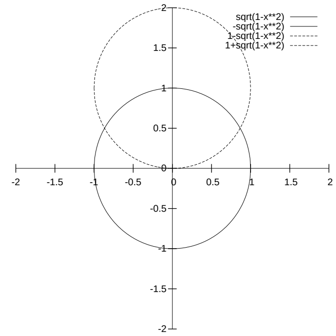
So the area is

$$\begin{aligned}
E_{b,a} &= \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} \left(\frac{1}{\sqrt{b}} \sqrt{1 - ax^2} - \left(-\frac{1}{\sqrt{b}} \sqrt{1 - ax^2} \right) \right) dx \\
&= 2 \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} \frac{1}{\sqrt{b}} \sqrt{1 - ax^2} dx \\
&= \frac{2}{\sqrt{b}} \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} \sqrt{1 - ax^2} dx \\
&= \frac{2}{\sqrt{b}} \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} \sqrt{1 - ax^2} dx & \left(u = \sqrt{a}x; \frac{du}{dx} = \sqrt{a} \right) \\
&= \frac{2}{\sqrt{b}\sqrt{a}} \int_{-1}^{1} \sqrt{1 - u^2} dx \\
&= \frac{\pi}{\sqrt{ab}},
\end{aligned}$$

which note makes a lot of sense: $ax^2 + by^2 = 1 \leftrightarrow (\sqrt{a}x)^2 + \left(\sqrt{b}y\right)^2 = 1$, so our ellipse is a circle scaled horizontally by $\frac{1}{\sqrt{a}}$ and vertically by $\frac{1}{\sqrt{b}}$.



Area of a crescent:
As seen from the earth, the disc of the sun has approximately the same radius as the disc of the moon. During a solar eclipse, the latter slides over the former. When the disc of the moon is centred at the edge of the disc of the sun, what proportion of the sun's disc is covered?



Intersection points:

$$\sqrt{1-x^2} = 1 - \sqrt{1-x^2} \leftrightarrow 2\sqrt{1-x^2} = 1 \leftrightarrow x = \pm\frac{\sqrt{3}}{2}$$

Area of covered area:

$$\begin{aligned}
C &= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-x^2} - \left(1 - \sqrt{1-x^2}\right)\right) dx \\
&= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(2\sqrt{1-x^2} - 1\right) dx \\
&= 2\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2} dx - \sqrt{3} \\
&= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(1 - \cos\left(2\theta\right)\right) d\theta - \sqrt{3} \\
&= \left[\theta\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \frac{1}{2}\left[\sin\left(2\theta\right)\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \sqrt{3} \\
&= \frac{2\pi}{3} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - \sqrt{3} \\
&= \frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \sqrt{3} \\
&= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \\
&= 0.391\pi
\end{aligned}$$

So 0.391 of the sun's disc is blocked.

Volumes

Example - Volume of a sphere: Consider a sphere of radius r, centred at the origin (0,0,0).

Chop it perpendicular to the x-axis into n slivers of equal width.

The volume of the sphere is the sum of the volumes of the slivers.

For large n, i.e. for thin slivers, each sliver is roughly a cylinder of width $\Delta_n = \frac{2r}{n}$. The radius depends on x: the i^{th} sliver has radius $\sqrt{r^2 - (x_i^*)^2}$ on its right face, where $x_i^* = -r + i\Delta_n$.

So we can estimate the volume of the i^{th} sliver as

$$\Delta_n \pi \left(\sqrt{r^2 - (x_i^*)^2}\right)^2 = \Delta_n \pi \left(r^2 - (x_i^*)^2\right)$$

So our estimate for the volume with n slivers is

$$V_n = \sum_{i=1}^{n} \Delta_n \pi \left(r^2 - (x_i^*)^2\right).$$

As $n \to \infty$, our estimates converge to the actual volume. So the volume of the sphere is

$$\begin{aligned}
V &= \lim_{n\to\infty} V_n \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_n \pi \left(r^2 - (x_i^*)^2\right) \\
&= \int_{-r}^{r} \pi \left(r^2 - x^2\right) dx \\
&= \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} \\
&= \pi \left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 - \frac{-r^3}{3}\right)\right) \\
&= \frac{4\pi r^3}{3}.
\end{aligned}$$

General formula: If a shape lies between $x = a$ and $x = b$, and the area of a cross-section perpendicular to the x-axis is a continuous function $A(x)$, then the volume is

$$\int_a^b A(x)\, dx.$$

Indeed, the argument above indicates that the volume is

$$\lim_{n\to\infty} \sum_{1}^{n} A(x_i^*)\, \Delta_n$$

where we divide $[a, b]$ into n intervals of equal width, and x_i^* is a point in the i^{th} interval and Δ_n is the width of an interval. But this is precisely the definition of the integral $\int_a^b A(x)\, dx$, which exists since A is continuous.

Volumes of revolution: We can think of the sphere as the result of taking the semicircle which is the region under the graph of $y = \sqrt{r^2 - x^2}$, and rotating it around the x axis.

The same trick for finding the volume works with any shape of that form:

Suppose $f(x)$ is continuous and non-negative on $[a, b]$. Consider the "solid of revolution" of the region under the graph of f between $x = a$ and $x = b$, being the space that shape passes through as you rotate it through 2π around the x-axis. Then the cross-sectional area at x is $\pi f(x)^2$, so the volume is given by

$$\pi \int_a^b f(x)^2 \, dx.$$

Example: Find the volume of the surface of revolution of e^x between 0 and 1.

Solution:

$$\begin{aligned} V &= \pi \int_0^1 (e^x)^2 \, dx \\ &= \pi \int_0^1 e^{2x} dx \\ &= \frac{\pi}{2} \left[e^{2x}\right]_0^1 dx \\ &= \frac{\pi}{2} \left(e^2 - 1\right) dx \end{aligned}$$